Introduction to twisted Alexander polynomials

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Feb. 17-19, 2015

Goal: how to compute twisted Alexander polynomial

Plan of lectures

- Alexander polynomial of a knot
  - Definition
  - Examples:trefoil knot, figure-eight knot
  - Relation to Reidemeister torsion, orders
- Twisted Alexander polynomial
  - History and definition
  - Examples:figure-eight knot, torus knots
  - Relations to
    - Reidemeister torsion
    - Order
- Applications
  - Detect the trivial knot
  - Fiberedness
  - epimorphisms between knot groups
  - L^2-invariants

Part 1: Alexander polynomial

Alexander polynomial of a knot

There are many definitions (many faces) of the classical Alexander polynomial.

- Seifert form on a Seifert surface
- Fox's free differential to a presentation of a knot group
- an order of the Alexander module (an infinite cyclic covering)
- Reidemeister torsion
- Burau representation of the braid group
- Obstruction to deform abelian representations into non commutative direction
- Skein relation
- Euler characteristic of knot Floer homology

How generalize to twisted cases ?

- Lin: Seifert form on a Seifert surface
- Wada:
  - Fox's free differential to a presentation of a knot group
  - Obstruction to deform abelian representations into non commutative direction
- Kirk-Livingston:an order of the Alexander module (an infinite cyclic covering)
- Kitano:Reidemeister torsion

In this lecture we follow the definition by Wada.

What is an Alexander polynomial of a knot ?

- $K \subset S^3$ a knot
- $G(K) = \pi_1(S^3 - K)$ its knot group
- $G(K) = \langle x_1, \cdots, x_n | r_1, \cdots, r_{n-1} \rangle$: (Wirtinger) presentation
- $A = \left(\frac{\partial r_i}{\partial x_j} \right) \in M(n \times (n-1); \mathbb{Z}[t, t^{-1}])$:Alexander matrix
- A_k: $(n-1) \times (n-1)$-matrix removed the k-th column

$\Delta_K(t) = \det(A_k)$:Alexander polynomial of K

In general situations the rational expression $\frac{\Delta_K(t)}{t-1}$ can be naturally defined and generalized to a twisted Alexander polynomial.

What is a twisted Alexander polynomial of a knot ?

- $K \subset S^3$ a knot
- $G(K) = \langle x_1, \cdots, x_n | r_1, \cdots, r_{n-1} \rangle$: Wirtinger presentation
- $\rho : G(K) \to GL(l; \mathbb{F})$ an l-dimensional linear representation over $\mathbb{F}$.
- $\mathbb{F} = \mathbb{Z}/p, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Q}(t)$
- $A_\rho \in M(ln \times l(n-1); \mathbb{F}[t, t^{-1}])$:twisted Alexander matrix
- $A_{\rho,k}$: $l(n-1) \times l(n-1)$-matrix removed the k-th column

$\Delta_{K,\rho}(t) = \frac{\det(A_{\rho,k})}{\det(\Phi(x_k - 1))} \in \mathbb{F}(t)$:twisted Alexander polynomial of (K, ρ)

Algebraic tools

- a group ring
- Fox's free differentials

Group ring

Definition

An integral group ring of a group G is a ring

$$\mathbb{Z}G = \{\text{a finite formal sum } \sum_{g\in G} n_g g \mid n_g \in \mathbb{Z}\}.$$

Here for any element $\sum_{g\in G} n_g g$ the number of $n_g \neq 0$ is finite.

- sum: $\displaystyle\sum_{g\in G} n_g g + \sum_{g\in G} m_g g = \sum_{g\in G}(n_g + m_g)g.$
- multiplication: $\displaystyle\sum_{g\in G} n_g g \cdot \sum_{g\in G} m_g g = \sum_{g\in G}\left(\sum_{h\in G} n_h \cdot m_{h^{-1}g}\right) g.$

Remark

- *The unit of* $\mathbb{Z}G$ *is* $1 = 1(\in \mathbb{Z}) \times 1(\in G)$.
- *We can also define* $\mathbb{Q}G$ *over* $\mathbb{Q}$, $\mathbb{R}G$ *over* $\mathbb{R}$, *and* $\mathbb{C}G$ *over* $\mathbb{C}$.
- *A group ring* $\mathbb{Z}G$ *is a commutative ring if and only if* G *is a commutative group*.

Example: $\mathbb{Z} = \langle t \rangle$

For any element of $\mathbb{Z}\mathbb{Z} = \mathbb{Z}\langle t \rangle$, it is a form of $\sum n_k t^k$. This is a Laurent polynomial of t. From now we always identify the group ring

$$\mathbb{Z}\mathbb{Z} = \mathbb{Z}\langle t \rangle \cong \mathbb{Z}[t, t^{-1}].$$

Fox's free differential

Let $F_n = \langle x_1, \cdots, x_n \rangle$ be the free group generated by $\{x_1, \cdots, x_n\}$.
Fox's free differentials are maps

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} : \mathbb{Z}F_n \to \mathbb{Z}F_n.$$

Characterization of Free differentials

1. They are linear over $\mathbb{Z}$
2. For any i, j, $\dfrac{\partial}{\partial x_j}(x_i) = \delta_{ij} = \begin{cases} 1 \ (i = j) \\ 0 \ (i \neq j) \end{cases}$.
3. (Leibniz rule): For any $g, g' \in F_n$,
$\dfrac{\partial}{\partial x_j}(gg') = \dfrac{\partial}{\partial x_j}(g) + g\dfrac{\partial}{\partial x_j}(g')$.

Lemma

The followings hold.

- $\dfrac{\partial}{\partial x_j}(1) = 0.$
- $\dfrac{\partial}{\partial x_j}(g^{-1}) = -g^{-1}\dfrac{\partial}{\partial x_j}(g)$ *for any* $g \in F_n$.
- $\dfrac{\partial}{\partial x_j}(x_j^k) = 1 + x_j + \cdots + x_j^{k-1}$ $(k > 0)$.
- $\dfrac{\partial}{\partial x_j}(x_j^k) = -(x_j^{-1} + \cdots + x_j^k)$ $(k < 0)$.
- $\dfrac{\partial}{\partial x_j}(g^k) = \dfrac{g^k - 1}{g - 1}\dfrac{\partial g}{\partial x_j}$ *for any* $g \in F_n, k > 0$.

For simplicity we write

$$\frac{\partial w}{\partial x_i} = \frac{\partial}{\partial x_i}(w).$$

The following formula is one algebraic version of a finite Taylor expansion in the group ring of a free group.

Proposition (Fundamental formula of free differentials)

For any $w \in \mathbb{Z}F_n$, it holds that

$$w - 1 = \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}(x_j - 1).$$

Apply to a knot in S^3

- $K \subset S^3$ a knot in S^3
- $N(K) \subset S^3$ an open tubular neighborhood of K
- $E(K) = S^3 \setminus N(K)$ an exterior of K, which is a compact 3-manifold with a torus boundary.
- $G(K) = \pi_1 E(K) \cong \pi_1(S^3 - K)$ the knot group of K.
- $G(K) = \langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1} \rangle$ a presentation of $G(K)$

We do not assume it is a Wirtinger presentation, only assume that

$$\begin{aligned}\text{deficiency} &= \text{the number of generators} - \text{the number of relators} \\ &= 1.\end{aligned}$$

By using the above fixed presentation, an epimorphism

$$F_n = \langle x_1, \ldots, x_n \rangle \twoheadrightarrow G(K)$$

is defined. Further we consider a ring homomorphism

$$\mathbb{Z}F_n \to \mathbb{Z}G(K)$$

induced from $F_n \to G(K)$.
The abelianization of $G(K)$

$$\alpha : G(K) \to G(K)/[G(K), G(K)] \cong \mathbb{Z} = \langle t \rangle.$$

Definition

The $(n-1) \times n$-matrix A defined by

$$A = \left(\alpha_* \left(\frac{\partial r_i}{\partial x_j} \right) \right) \in M\left((n-1) \times n; \mathbb{Z}[t, t^{-1}]\right)$$

is called the Alexander matrix of $G(K) = \langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1} \rangle$.
Here $\alpha_* : \mathbb{Z}G(K) \to \mathbb{Z}\langle t \rangle = \mathbb{Z}[t, t^{-1}]$.

- A_k:$(n-1) \times (n-1)$-matrix obtained by removing the k-th column from A.

Lemma

There exists $k \in \{1, \cdots, n\}$ *s.t.* $\alpha_*(x_k) - 1 \neq 0 \in \mathbb{Z}[t, t^{-1}]$.

Proof.

If $\alpha(x_k) = 1$ for any k, then $\alpha : G(K) \to \mathbb{Z}$ is the trivial homomorphism, not an epimorphism. □

Remark

The condition $\alpha(x_k) - 1 \neq 0$ *is corresponding to the vanishing of* $H_0(E(K); \mathbb{Q}(t)_\alpha)$. *Here* $\alpha : G(K) \to \langle t \rangle \subset GL(1; \mathbb{Q}(t))$.

Lemma

For any k, l,

$$(\alpha_*(x_l) - 1)\, detA_k = \pm\,(\alpha_*(x_k) - 1)\, detA_l$$

From this lemma, we can consider

$$\frac{\det A_k}{\alpha_*(x_k) - 1}$$

as an invariant of $G(K)$.

Proof.

We can assume $k = 1, l = 2$.
For $r_i = 1 \in \mathbb{Z}G(K)$, we apply the fundamental formula,

$$0 = r_i - 1 = \sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j}(x_j - 1).$$

Apply α_* to both sides,

$$\sum_{j=1}^{n} \alpha_* \left(\frac{\partial r_i}{\partial x_j} \right) (\alpha_*(x_j) - 1) = 0$$

Then

$$(\alpha_*(x_1)-1)\alpha_*\left(\frac{\partial r_i}{\partial x_1}\right) = -\sum_{j=2}^{n}\alpha_*\left(\frac{\partial r_i}{\partial x_j}\right)(\alpha_*(x_j)-1)$$

- A_2:removed the second column from A.
- $\tilde{A}_2$:replaced the first column $\alpha_*\left(\frac{\partial r_i}{\partial x_1}\right)$ to $(\alpha_*(x_1)-1)\alpha_*\left(\frac{\partial r_i}{\partial x_1}\right)$ in A_2.

Take its determinant;

$$\det\tilde{A}_2 = \begin{vmatrix} (\alpha_*(x_1)-1)\,\alpha_*\left(\frac{\partial r_1}{\partial x_1}\right) & \alpha_*\left(\frac{\partial r_1}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_1}{\partial x_n}\right) \\ \vdots & \dots\dots\dots\dots & \dots\dots & \vdots \\ (\alpha_*(x_1)-1)\,\alpha_*\left(\frac{\partial r_{n-1}}{\partial x_1}\right) & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_n}\right) \end{vmatrix}$$

$$= (\alpha_*(x_1)-1)\det A_2$$

On the other hand,

$$\det \tilde{A}_2 = \begin{vmatrix} -\displaystyle\sum_{j=2}^{n} \alpha_*\left(\frac{\partial r_1}{\partial x_j}\right)(\alpha_*(x_j)-1) & \dots & \alpha_*\left(\frac{\partial r_1}{\partial x_n}\right) \\ \vdots & \dots & \vdots \\ -\displaystyle\sum_{j=2}^{n} \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_j}\right)(\alpha_*(x_j)-1) & \dots & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_n}\right) \end{vmatrix}$$

$$= -\sum_{j=2}^{n}(\alpha_*(x_j)-1) \begin{vmatrix} \alpha_*\left(\frac{\partial r_1}{\partial x_j}\right) & \alpha_*\left(\frac{\partial r_1}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_1}{\partial x_n}\right) \\ \vdots & \dots\dots\dots\dots\dots\dots & & \vdots \\ \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_j}\right) & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_n}\right) \end{vmatrix}$$

$$
= -(\alpha_*(x_2) - 1) \begin{vmatrix} \alpha_*\left(\frac{\partial r_1}{\partial x_2}\right) & \alpha_*\left(\frac{\partial r_1}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_1}{\partial x_n}\right) \\ \vdots & \dots\dots\dots & \dots\dots\dots & \vdots \\ \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_2}\right) & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_n}\right) \end{vmatrix}
$$

$$
= -(\alpha_*(x_2) - 1)\det A_1
$$

Therefore

$$
(\alpha_*(x_1) - 1)\det A_2 = -(\alpha_*(x_2) - 1)\det A_1.
$$

Proposition

Up to $\pm t^s$ $(s \in \mathbb{Z})$*, the rational expression*

$$\frac{\det A_k}{\alpha_*(x_k) - 1}$$

is independent of the choice of a presentation of $G(K)$*. Namely it is an invariant of a group* $G(K)$*.*

Proof.

It can be directly checked for Tietze transformations. $\square$

Tietze transformations

Let G be a group and $\langle x_1, \cdots, x_k | r_1, \cdots, r_l \rangle$ a presentation of G.

Theorem (Tietze)

Any presentation $G = \langle x_1, \cdots, x_k | r_1, \cdots, r_l \rangle$ can be transformed to any other presentation of G by an application of a finite sequence of of the following two type operations and their inverses:

- *To add a consequence r of the ralators $r_1, \cdots, r_l$ to the set of relators. The resulting presenation is $\langle x_1, \cdots, x_k | r_1, \cdots, r_l, r \rangle$.*
- *To add a new generator x and a new relator xw^{-1} where w is any word in $x_1, \cdots, x_k$. The resulting presentation is $\langle x_1, \cdots, x_k, x | r_1, \cdots, r_l, xw^{-1} \rangle$.*

For a knot K, we can take some special presentation of $G(K)$ from a regular diagram on the plane. Such a presentation is called a Wirtinger presentation of $G(K)$ (we omit to explain precisely).

Definition

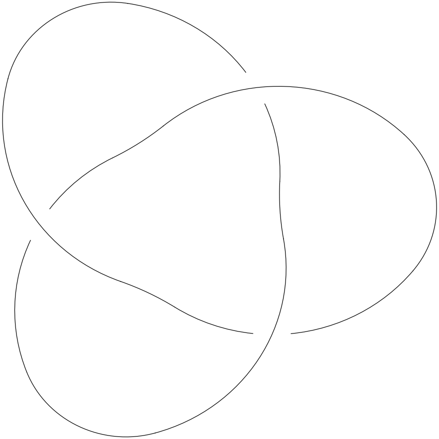
If we take a Wirtinger presentation of $G(K)$ (in this case any $\alpha(x_j) = t$), the denominator is always $t - 1$. Then the numerator is an invariant of $G(K)$ up to $\pm t^s$. This is the Alexander polynomial $\Delta_K(t)$ of K.

$$\frac{\det A_k}{t-1} = \frac{\Delta_K(t)}{t-1}.$$

Remark

Alexander polynomial is well-defined up to $\pm t^s$.

Torefoile knot 3_1



3_1

$$G(3_1) = \langle x, y \mid r = xyx(yxy)^{-1} \rangle$$

By applying α, the relator r goes to

$$\begin{aligned}
r &= xyx(yxy)^{-1} \\
&= xyxy^{-1}x^{-1}y^{-1} \\
&\mapsto \alpha(x)\alpha(y)\alpha(x)\alpha(y)^{-1}\alpha(x)^{-1}\alpha(y)^{-1} \\
&= \alpha(x)\alpha(y)^{-1} \in G(3_1)/[G(3_1), G(3_1)],
\end{aligned}$$

then we get

$$\alpha(x)\alpha(y)^{-1} = 1 \in G(3_1)/[G(3_1), G(3_1)].$$

Hence the abelianization is given by

$$\alpha : G(3_1) \ni x, y \mapsto t \in \langle t \rangle.$$

Here

$$\begin{aligned}
\frac{\partial}{\partial x}(r) &= \frac{\partial}{\partial x}(xyx(yxy)^{-1}) \\
&= \frac{\partial}{\partial x}(xyx) - xyx(yxy)^{-1}\frac{\partial}{\partial x}(yxy) \\
&= \frac{\partial}{\partial x}(xyx) - r\frac{\partial}{\partial x}(yxy).
\end{aligned}$$

In $\mathbb{Z}G(3_1)$, and further in $\mathbb{Z}[t, t^{-1}]$,

$$r = 1.$$

Then in $\mathbb{Z}[t, t^{-1}]$,

$$\begin{aligned}\frac{\partial}{\partial x}\left((xyx(yxy)^{-1}\right) &= \frac{\partial}{\partial x}(xyx) - \frac{\partial}{\partial x}(yxy) \\ &= \frac{\partial}{\partial x}(xyx - yxy)\end{aligned}$$

Therefore we can compute free differentials for $xyx - yxy$ instead of $r = xyx(yxy)^{-1}$. Here

$$\begin{aligned}\frac{\partial}{\partial x}(xyx - yxy) &= \frac{\partial}{\partial x}(xyx) - \frac{\partial}{\partial x}(yxy) \\ &= 1 + xy - y \\ &\mapsto \alpha_*(1 + xy - y) \\ &= t^2 - t + 1 \in \mathbb{Z}[t, t^{-1}]\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial}{\partial y}(xyx - yxy) &= \frac{\partial}{\partial y}(xyx) - \frac{\partial}{\partial y}yxy \\ &= x - 1 - yx \\ &\mapsto \alpha_*(x - 1 - yx) \\ &= -(t^2 - t + 1) \in \mathbb{Z}[t, t^{-1}]\end{aligned}$$

Hence

$$A = \begin{pmatrix}(t^2 - t + 1) & -(t^2 - t + 1)\end{pmatrix},$$

$$\begin{aligned}\frac{\det A_2}{t - 1} &= -\frac{\det A_1}{t - 1} \\ &= \frac{t^2 - t + 1}{t - 1}.\end{aligned}$$

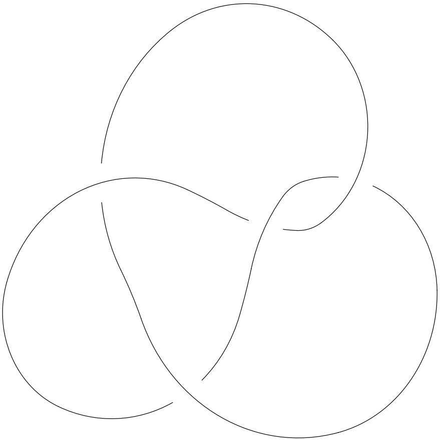
We change this presentation to $\langle x, y, z \mid xyx(yxy)^{-1}, xyz^{-1} \rangle$. Then its Alexander matrix is

$$A = \begin{pmatrix} (t^2 - t + 1) & -(t^2 - t + 1) & 0 \\ 1 & t & -1 \end{pmatrix}$$

From this Alexander matrix, we have

$$\begin{aligned} \frac{\det A_1}{t-1} &= \frac{t^2 - t + 1}{t-1}, \\ \frac{\det A_2}{t-1} &= -\frac{t^2 - t + 1}{t-1}, \\ \frac{\det A_3}{t^2-1} &= \frac{t(t^2 - t + 1) + (t^2 - t + 1)}{t^2-1} \\ &= \frac{t^2 - t + 1}{t-1}. \end{aligned}$$

Figure-eight knot 4_1



4_1

From the diagram of $G(4_1)$, we can get

$$
\begin{aligned}
&\langle\ x_1, x_2, x_3, x_4 \mid x_4 = x_1 x_3 x_1^{-1}, x_2 = x_3 x_1 x_3^{-1}, x_2 x_1 x_2^{-1} x_4^{-1}\ \rangle \\
=&\langle\ x_1, x_3 \mid (x_3 x_1 x_3^{-1}) x_1 (x_3 x_1 x_3^{-1})^{-1} (x_1 x_3 x_1^{-1})^{-1}\ \rangle \\
=&\langle\ x_1, x_3 \mid x_3 x_1 x_3^{-1} x_1 x_3 x_1^{-1} x_3^{-1} x_1 x_3^{-1} x_1^{-1}\ \rangle \\
=&\langle\ x_1, x_3 \mid x_1^{-1} x_3 x_1 x_3^{-1} x_1 x_3 x_1^{-1} x_3^{-1} x_1 x_3^{-1}\ \rangle \text{ (conjugated by } x_1^{-1}) \\
=&\langle\ x, y \mid w x w^{-1} = y\ \rangle
\end{aligned}
$$

where $x = x_1, y = x_3, w = x^{-1} y x y^{-1}$.

Further the abelianization

$$\alpha : G(4_1) \to \langle t \rangle$$

is given by $\alpha(x) = \alpha(y) = t$.
Then we have

$$\begin{aligned}
\frac{\partial}{\partial x}(wxw^{-1}y^{-1}) &= \frac{\partial w}{\partial x} + w\frac{\partial x}{\partial x} - wxw^{-1}\frac{\partial w}{\partial x} \\
&= (1-y)\frac{\partial w}{\partial x} + w \\
&\mapsto \alpha_* \left((1-y)\frac{\partial w}{\partial x} \right) + \alpha_*(w) \\
&= (1-t)\alpha_* \left(\frac{\partial w}{\partial x} \right) + 1.
\end{aligned}$$

Here

$$
\begin{aligned}
\alpha_*\left(\frac{\partial w}{\partial x}\right) &= \alpha_*\left(\frac{\partial}{\partial x}(x^{-1}yxy^{-1})\right) \\
&= \alpha_*(-x^{-1} + x^{-1}y) \\
&= -t^{-1} + 1.
\end{aligned}
$$

Then

$$
\begin{aligned}
\alpha_*\left(\frac{\partial}{\partial x}(wxw^{-1}y^{-1})\right) &= (1-t)(-t^{-1}+1)+1 \\
&= -t^{-1}+1+1-t(-t^{-1}+1) \\
&= -t^{-1}+1+1+1-t \\
&= -t^{-1}+3-t
\end{aligned}
$$

Similarly

$$\begin{aligned}\alpha_*\left(\frac{\partial}{\partial y}(wxw^{-1}y^{-1})\right) &= \alpha_*\left((1-y)\frac{\partial w}{\partial x}-1\right)\\ &= (1-t)(t^{-1}-1)-1\\ &= t^{-1}-3+t.\end{aligned}$$

Hence

$$A=\begin{pmatrix}-t^{-1}+3-t & t^{-1}-3+t\end{pmatrix}.$$

$$\frac{\det A_1}{\alpha_*(x_1)-1} = \frac{t^{-1}-3+t}{t-1}$$
$$= -\frac{1}{t}\frac{(-t^2+3t-1)}{t-1}.$$

$$\frac{\det A_2}{\alpha_*(x_2)-1} = -\frac{t^{-1}-3+t}{t-1}$$
$$= \frac{1}{t}\frac{(-t^2+3t-1)}{t-1}.$$

Up to $\pm t^s$,

$$\Delta_K(t) = -t^2+3t-1.$$

Remark

In the computation of free differentials, by applying α_*,

$$\alpha_*\left(\frac{\partial}{\partial x}(wxw^{-1}y^{-1})\right) = \alpha_*\left(\frac{\partial}{\partial x}(wxw^{-1} - y)\right) = \alpha_*\left(\frac{\partial}{\partial x}(wx - yw)\right),$$

then we can apply free differentials for $wxw^{-1} - y$, *or* $wx - yw$ *instead of* $wxw^{-1}y^{-1}$.

The case of deficiency ≤ 0

If the deficiency is less than or equal to 0,

$$G = \langle\, x_1, \ldots, x_n \mid r_1, \ldots, r_m \,\rangle \; (n \leq m)$$

A is an $m \times n$-matrix, and A_k is an $m \times (n-1)$-matrix. In the case, A_k is not a square matrix. Then we consider all $(n-1) \times (n-1)$-minors and take its gcd, which is written by Q_k. Now we have the following.

Proposition

For any k s.t. $\alpha_*(x_k) - 1 \neq 0$, the following expression

$$\frac{Q_k}{\alpha_*(x_k) - 1}$$

is independent of the choice of presentation up to $\pm t^s (s \in \mathbb{Z})$ and gives an invariant go G.

Relation to Reidemeister torsion

- $K \subset S^3$ a knot
- $E(K) = S^3 - N(K)$ its exterior of K.
- $\alpha : G(K) \to T = \langle t \rangle \subset GL(1; \mathbb{Q}(t))$ an 1-dimensional representation over $\mathbb{Q}(t)$.
- $\mathbb{Q}(t)$ the rational function field over $\mathbb{Q}$
- $\tau_\alpha(E(K)) \in \mathbb{Q}(t)$ Reidemeister torsion for $(E(K), \alpha)$.

Theorem (Milnor)

$$\frac{\Delta_K(t)}{t-1} = \tau_\alpha(E(K)).$$

What is Reidemeister torsion ?

- X a compact CW-complex
- $\rho : \pi_1(X) \to GL(V)$ a linear representation over $\mathbb{F}$
- $C_*(X; V_\rho) = C_{even} \oplus C_{odd}$ with $\partial : C_{even} \oplus C_{odd} \to C_{even} \oplus C_{odd}$
- Assume $H_*(X; V_\rho) = 0$
- Fix bases $\mathbf{c}_{even}$ and $\mathbf{c}_{odd}$ and take bases $\mathbf{b}_{even}$, $\mathbf{b}_{odd}$.
- Get other bases of C by $\mathbf{b}_{even}$ and $\mathbf{b}_{odd}$.
- $\tau_\rho(X) = \dfrac{\det((\mathbf{b}_{even}, \mathbf{b}_{odd}) \to \mathbf{c}_{odd})}{\det((\mathbf{b}_{even}, \mathbf{b}_{even}) \to \mathbf{c}_{even})} \in \mathbb{F}$

Remark

We have to twist homologies by ρ for the acyclicity.

torsion of a chain complex

a chain complex C_*

$$0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} C_{m-2} \longrightarrow \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0.$$

Definition

- $Z_q = \ker \partial_q \subset C_q$
- $B_q = Im\partial_{q+1} \subset Z_q \subset C_q$

Because $0 \longrightarrow Z_q \longrightarrow C_q \xrightarrow{\partial_q} B_{q-1} \longrightarrow 0$ is exact, then we have

$$C_q \cong Z_q \oplus B_{q-1}$$

(not canonical).

Definition

C_* is called to be acyclic if

$$B_q = Z_q \text{ (that is, } H_q(C_*) = 0)$$

for any $q = 0, 1, \ldots, m$.

Remark

a chain complex C_* is acyclic if and only if C_* is an exact sequence.

- We assume a basis $\mathbf{c}_q$ of C_q is given for any q.
- We also take a basis $\mathbf{b}_q$ on the q-th boundary B_q for any q.

On this exact sequence

$$0 \longrightarrow Z_q \longrightarrow C_q \xrightarrow{\partial_q} B_{q-1} \longrightarrow 0$$

by taking a lift $\tilde{\mathbf{b}}_{q-1}$ of $\mathbf{b_{q-1}}$, $(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1})$ is a basis on C_q. We have

$$C_q \cong B_q \oplus B_{q-1}$$

Let $\mathbf{b} = \{b_1, \cdots, b_n\}, \mathbf{c} = \{c_1, \cdots, c_n\}$ be two bases of a vector space V over $\mathbb{F}$.
There exists a non-singular matrix $P = (p_{ij})$ s.t. $b_j = \sum p_{ji} c_i$.

Definition

P is called the transformation matrix from $\mathbf{c}$ to $\mathbf{b}$ denoted by $(\mathbf{b}/\mathbf{c})$. Its determinant $\det P$ is denoted by $[\mathbf{b}/\mathbf{c}]$.

Under definitions

- $\left(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q\right)$: the transformation matrix from $\mathbf{c}_q$ to $(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1})$.
- $\left[\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q\right]$: its determinant $\det\left(\mathbf{b_q}, \tilde{\mathbf{b}}_{\mathbf{q}-1}/\mathbf{c_q}\right)$.

Lemma

The determinant $[\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q]$ *is independent on the choice of a lift* $\tilde{\mathbf{b}}_{q-1}$. *Hence we can simply write* $[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]$ *to it.*

Proof.

Assume $\hat{\mathbf{b}}_{q-1}$ is another lift of $\mathbf{b}_{q-1}$ on C_q. Here

$$0 \longrightarrow Z_q \longrightarrow C_q \longrightarrow B_{q-1} \longrightarrow 0$$

is an exact sequence, then a difference between any vector of $\hat{\mathbf{b}}_{q-1}$ and the corresponding one of $\tilde{\mathbf{b}}_{q-1}$ belongs to $Z_q = B_q$. Then by the definition of det,

$$\left[\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q\right] = \left[\mathbf{b}_q, \hat{\mathbf{b}}_{q-1}/\mathbf{c}_q\right]$$

□

Definition

The torsion $\tau(C_*)$ of a chain complex C_* is defined by

$$\tau(C_*) = \frac{\prod_{q:odd}[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]}{\prod_{q:even}[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]} \in \mathbb{F} \setminus \{0\}.$$

Lemma

The torsion $\tau(C_*)$ is independent of the choice of $\mathbf{b}_q$.

Proof.

Assume $\mathbf{b_q'}$ is another basis of B_q.
In the definition of $\tau(C_*)$, the difference between $\mathbf{b_q}$ and $\mathbf{b_q'}$ is related to the followings only two parts:

$$\left[\mathbf{b_q'}, \mathbf{b_{q-1}}/\mathbf{c_q}\right] = \left[\mathbf{b_q}, \mathbf{b_{q-1}}/\mathbf{c_q}\right]\left[\mathbf{b_q'}/\mathbf{b_q}\right]$$

$$\left[\mathbf{b_{q+1}}, \mathbf{b'_q}/\mathbf{c_{q+1}}\right] = \left[\mathbf{b_{q+1}}, \mathbf{b_q}/\mathbf{c_{q+1}}\right]\left[\mathbf{b_q'}/\mathbf{b_q}\right]$$

Since $\left[\mathbf{b_q'}/\mathbf{b_q}\right]$ appears in the both of the denominator and the numerator of the definition, they can be cancelled. □

Example

$$C_* : 0 \to C_4 \to C_3 \to C_2 \to C_1 \to C_0 \to 0.$$

As $\mathbf{b}_4$ and $\mathbf{b}_{-1}$ are zero, then

$$\begin{aligned}\tau(C_*) &= \frac{[\mathbf{b}_4, \mathbf{b}_3/\mathbf{c}_4][\mathbf{b}_2, \mathbf{b}_1/\mathbf{c}_2][\mathbf{b}_0, \mathbf{b}_{-1}/\mathbf{c}_0]}{[\mathbf{b}_3, \mathbf{b}_2/\mathbf{c}_3][\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1]} \\ &= \frac{[\mathbf{b}_3/\mathbf{c}_4][\mathbf{b}_2, \mathbf{b}_1/\mathbf{c}_2][\mathbf{b}_0/\mathbf{c}_0]}{[\mathbf{b}_3, \mathbf{b}_2/\mathbf{c}_3][\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1]}\end{aligned}$$

In this case the number of the denominator and the number of numerator are not the same.

Example

$$C_* : 0 \to C_3 \to C_2 \to C_1 \to C_0 \to 0.$$

As $\mathbf{b}_3$ and $\mathbf{b}_{-1}$ are zero, then

$$\begin{aligned}\tau(C_*) &= \frac{[\mathbf{b}_2, \mathbf{b}_1/\mathbf{c}_2][\mathbf{b}_0, \mathbf{b}_{-1}/\mathbf{c}_0]}{[\mathbf{b}_3, \mathbf{b}_2/\mathbf{c}_3][\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1]} \\ &= \frac{[\mathbf{b}_2, \mathbf{b}_1/\mathbf{c}_2][\mathbf{b}_0/\mathbf{c}_0]}{[\mathbf{b}_2/\mathbf{c}_3][\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1]}\end{aligned}$$

Only $\mathbf{b}_3$ appears once, but this is also zero. In this case the number of the denominator and the number of numerator are same.

Mayer-Vietoris argument for a torsion invariant

Lemma

Assume

$$0 \to C'_* \to C_* \to C''_* \to 0$$

is an exact sequence of chain complexes and a basis $(\mathbf{c}'_i, \mathbf{c}''_i)$ *of* C_* *as a union of bases on others. If two of* C'_*, C_*, C''_* *are acyclic, then the third one is also acyclic and*

$$\tau(C_*) = \pm\tau(C'_*)\tau(C''_*).$$

Why does $\pm$ appear in the right hand side?
To define the torsions we use

- $C'_* \cong Z'_* \oplus B'_*$, $C_* \cong Z_* \oplus B_*$, $C''_* \cong Z''_* \oplus B''_*$.

On the other hand, to get this formula, we use

- $C_* \cong C'_* \oplus C''_* \cong Z'_* \oplus B'_* \oplus Z''_* \oplus B''_*$.

Geometric settings

We take its universal cover

$$\tilde{E}(K) \to E(K).$$

We assume $G(K)$ acts on $\tilde{E}(K)$ from the right. Then

$$C_*(E(K); \mathbb{Q}(t)_\alpha) = C_*(\tilde{E}(K); \mathbb{Z}) \otimes_{\pi_1(X)} \mathbb{Q}(t).$$

By using this representation α, Reidemeister torsion of $E(K)$

$$\tau_\alpha(E(K)) = \tau(C_*(E(K); \mathbb{Q}(t)_\alpha)) \in \mathbb{Q}(t) \setminus \{0\}$$

can be defined up to $\pm t^s$.

Alexander polynomial=Reidemeister torsion

Theorem (Milnor)

$$\tau_\alpha(E(K)) = \frac{\Delta_K(t)}{t-1}.$$

From this theorem, some properties of Reidemeister torsion induce properties of Alexander polynomial.

Theorem (Seifert)

Up to $\pm t^s$, $\Delta_K(t^{-1}) = \Delta_K(t)$.

This can be obtained from the following.

Theorem (Milnor)

$$\tau_\alpha(E(K)) = \pm\overline{\tau_\alpha(E(K))}.$$

where

$$\bar{\ } : \mathbb{Z}[t, t] \ni f(t) \mapsto \overline{f(t)} = f(t^{-1}) \in \mathbb{Z}[t, t^{-1}].$$

We also have the following from the property of Reidemeister torsion.

Theorem (Fox-Milnor)

If K is a slice knot, then $\Delta_K(t) = f(t)f(t^{-1})$ where $f(t) \in \mathbb{Z}[t]$.

Definition

$K \subset S^3$ be a slice knot if there exists an embedded disk $B \subset B^4$ s.t. $\partial B = K \subset S^3 = \partial B^4$.

Part 2:Twisted Alexander polynomial

From Alexander polynomial to twisted Alexander polynomial

Alexander polynomial can be defined by

- Fox's free differential
- Reidemeister torsion

Here we want to mention two more thing;

- an order of $H_1(E(K); \mathbb{Q}[t, t^{-1}]_\alpha)$
- an obstruction to deform abelian representation

It is seen

$$\begin{aligned} &H_1(E(K); \mathbb{Q}[t, t^{-1}]_\alpha) \\ \cong &H_1(E(K)_\infty; \mathbb{Q}) \text{ as a } \mathbb{Q}[t, t^{-1}] - \text{module} \end{aligned}$$

because $\alpha : G(K) \to \mathbb{Z}$ is corresponding to $\mathbb{Z}$-cover $E(K)_\infty \to E(K)$.

Let M be a finitely generated $\mathbb{Q}[t, t^{-1}]$-module without free parts. From the structure theorem of a finitely generated module over a princeipal ideal domain, we have

$$M \cong \mathbb{Q}[t, t^{-1}]/(p_1) \oplus \cdots \oplus \mathbb{Q}[t, t^{-1}]/(p_k)$$

where $p_1, \cdots, p_k \in \mathbb{Q}[t, t^{-1}]$ s.t.

$$\mathbb{Q}[t, t^{-1}] \supsetneq (p_1) \supset (p_2) \supset \cdots \supset (p_k) \neq (0).$$

Definition

The order ideal of M is defined by

$$\mathrm{ord}(M) = (p_1 \cdots p_k) \subset \mathbb{Q}[t, t^{-1}].$$

Alexander polynomial = an order

Proposition

- $ord(H_1(E(K); \mathbb{Q}[t, t^{-1}]_\alpha)) = (\Delta_K(t))$.
- $ord(H_0(E(K); \mathbb{Q}[t, t^{-1}]_\alpha)) = (t - 1)$.

Then if we consider a twisted homology $H_*(E(K); \mathbb{Q}[t, t^{-1}]^l_{\rho\otimes\alpha})$, we get

- orders of $H_*(E(K); \mathbb{Q}[t, t^{-1}]^l_{\rho\otimes\alpha})$
- a generalization of the Alexander polynomial as generators of order ideals.

In this lecture we follow the Wada's definition, which is the ratio of two generators of order ideals.

Obstruction

- $G(K) = \langle x_1, \cdots, x_2 | r_1, \cdots, r_{n-1} \rangle$ Wirtinger presentation
- $\alpha_a = \alpha|_{t=a} : G(K) \ni x_i \mapsto a \in \mathbb{C}$ $(a \neq 0)$
- $\rho_a(x_i) = \begin{pmatrix} a & b_i \\ 0 & 1 \end{pmatrix} \in GL(2; \mathbb{C})$

If all $b_1, \cdots, b_n = 0$, clearly ρ_a gives a representation

$$\rho_a : G(K) \to GL(2; \mathbb{C}).$$

However it is also abelian.

Problem

When ρ_a can be extended as a non abelian representation ?

Alexander polynomial = Obstruction

Theorem (de Rham)

ρ_a gives a non abelian representation if and only if $\Delta_K(a) = 0$.

We can say Alexander polynomial is an obstruction to deform an 1-dimensional abelian representation in $\mathbb{C} \rtimes \mathbb{C}^*$.

Remark

This is one motivation for Wada to define twisted Alexander polynomial. That is, how we can generalize an obstruction for a higher dimensional representation.

What is a twisted Alexander polynomial ?

- $K \subset S^3$ a knot
- $G(K)$ its knot group
- $\alpha : G(K) \to \mathbb{Z}$ the abelianization
- $\rho : G(K) \to GL(l; R)$ a representation
- R a Euclidean domain (Euclidean division algorithm)
- $\Delta_{K,\rho}(t)$ twisted Alexander polynomial of (K, ρ)
- $\Delta_{K,\rho}(t)$ is a rational expression and well-defined up to $\pm\epsilon t^s$ where $\epsilon \in R$ is a unit.

For simplicity

We consider a representation of $G(K)$ in

- 2-dimensional unimodular group
- over a field.

From these assumption $\Delta_{K,\rho}(t)$ is well-defined up to t^s

History of Twisted Alexander polynomial

In the debut of twisted Alexander polynomial

- X. S. Lin, Representations of knot groups and twisted Alexander polynomials, Acta Mathematica Sinica, English Series, **17** (2001), No.3, pp. 361–380
  - for a knot by using a Seifert surface.
- M. Wada, *Twisted Alexander polynomial for finitely presentable groups*, Topology **33** (1994), 241–256.
  - for a finitely presentable group with an epimorphism onto $\mathbb{Z}$.

Remark

We follow the definition due to Wada, because

- *it is most computable*
- *it is equal to Reidemeister torsion*

- $K \subset S^3$ a knot K
- $G(K) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle$ its knot group with a deficieny one presentation.
- Let $F_n = \langle x_1, \ldots, x_n \rangle$ denote the free group.
- $\alpha : G(K) \to \mathbb{Z} = \langle t \rangle$ be the abelianization
- $\rho : G(K) \to SL(2; \mathbb{F})$ a representation.

- $M(2;\mathbb{F})$ is the matrix algebra of 2×2 matrices over $\mathbb{F}$.
- $\rho_* : \mathbb{Z}G(K) \to \mathbb{Z}SL(2;\mathbb{F}) \cong M(2;\mathbb{F})$ a ring homomorphism induced by ρ
- $\alpha_* : \mathbb{Z}G(K) \to \mathbb{Z}\mathbb{Z} = \mathbb{Z}\langle t \rangle \cong \mathbb{Z}[t, t^{-1}]$ a ring homomorphism induced by α
- $\rho_* \otimes \alpha_* : \mathbb{Z}G(K) \to M(2;\mathbb{F}) \otimes \mathbb{Z}[t, t^{-1}] \cong M\left(2;\mathbb{F}[t, t^{-1}]\right)$ is an induced ring homomorphism.

- $\Phi : \mathbb{Z}F_n \to M\left(2; \mathbb{F}[t, t^{-1}]\right)$ the composite of
    - $\mathbb{Z}F_n \to \mathbb{Z}G(K)$ induced by the presentation
    - $\rho_* \otimes \alpha_* : \mathbb{Z}G(K) \to M\left(2; \mathbb{F}[t, t^{-1}]\right)$.
- $$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} : \mathbb{Z}F_n \to \mathbb{Z}F_n$$

  the Fox's free differentials.

Definition

The $(n-1) \times n$ matrix A_ρ whose (i,j) component is the 2×2 matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M\left(2; \mathbb{F}[t, t^{-1}]\right),$$

This matrix is called the twisted Alexander matrix of $G(K) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle$ associated to ρ.

Remark

$$\begin{aligned} A_\rho &\in M\left((n-1) \times n; M\left(2; \mathbb{F}[t, t^{-1}]\right)\right) \\ &= M\left(2(n-1) \times 2n; \mathbb{F}[t, t^{-1}]\right). \end{aligned}$$

- $A_{\rho,k}$: the $(n-1)\times(n-1)$ matrix obtained from A_ρ by removing the kth column.
-

$$A_{\rho,k} \in M\left((n-1)\times(n-1); M\left(2;\mathbb{F}[t,t^{-1}]\right)\right)$$
$$= M\left(2(n-1)\times 2(n-1);\mathbb{F}[t,t^{-1}]\right).$$

Lemma

There exists k s.t. $\det\Phi(x_k-1)\neq 0$.

Lemma

$\det A_{\rho,k}\cdot\det\Phi(x_j-1)=\det A_{\rho,j}\cdot\det\Phi(x_k-1)$ *for any* j,k.

Remark

The above holds up to ± 1 *for an odd-dimensional representation.*

From the above two lemmas, we can define the twisted Alexander polynomial of $G(K)$ associated $\rho : G(K) \to SL(2;\mathbb{F})$ to be a rational expression as follows.

Definition

$$\Delta_{K,\rho}(t) = \frac{\det A_{\rho,k}}{\det \Phi(x_k - 1)}$$

for any k s.t $\det \Phi(x_k - 1) \neq 0$.

Remark

Up to t^s ($s \in \mathbb{Z}$), this is an invariant of $(G(K), \rho)$. Namely, it does not depend on the choices of a presentation. Hence we can consider it as a knot invariant.

Remark

The numerator of the twisted Alexander polynomial is also called the twisted Alexander polynomial and written as $\Delta_{K,\rho}(t)$.

In general, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ depends on ρ. However the following proposition can be proved easily.

Definition

Two representations ρ and ρ' are conjugate if there exists $P \in SL(2;\mathbb{F})$ s.t. $\rho(x) = P\rho'(x)P^{-1}$ for any $x \in G(K)$.

Proposition

If ρ *and* ρ' *are conjugate, then* $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$ *up to* t^s.

Example:trivial knot

- If K is trivial, we take the presentation as

$$G(K) = \langle x \rangle.$$

- Any representation $\rho : G(K) \to SL(2; \mathbb{C})$ is given by just one matrix $\rho(x)$.
-

$$\begin{aligned}\Delta_{K,\rho}(t) &= \frac{1}{\det(t\rho(x) - I)} \\ &= \frac{1}{(\lambda_1 t - 1)(\lambda_2 t - 1)}\end{aligned}$$

where λ_1, λ_2 are the eigenvalue of $\rho(x)$.

Example:trivial represenatation

- $\rho : G(K) \ni x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL(2;\mathbb{C})$ a 2 dimensional trivial representation
- $\alpha : G(K) \to \mathbb{Z} = \langle t \rangle$

$$\rho \otimes \alpha = \alpha \oplus \alpha : G(K) \ni x \mapsto \begin{pmatrix} \alpha(x) & 0 \\ 0 & \alpha(x) \end{pmatrix} \in GL(2;\mathbb{C}(t))$$

Hence we get

$$\begin{aligned} \Delta_{K,\alpha\oplus\alpha}(t) &= \frac{\Delta_K(t)}{t-1} \cdot \frac{\Delta_K(t)}{t-1} \\ &= \left(\frac{\Delta_K(t)}{t-1} \right)^2 \end{aligned}$$

Remark

In general for $\rho = \rho_1 \oplus \rho_2$, then $\Delta_{K,\rho}(t) = \Delta_{K,\rho_2}(t) \cdot \Delta_{K,\rho_2}(t)$.

Growth piriod in 1990's

- B. Jiang and S. Wang, Twisted topological invariants associated with representations, in Topics in knot theory (Erzurum, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 399, Kluwer Acad. Publ., Dordrecht (1993).
- T. Kitano, Twisted Alexander polynomial and Reidemeister torsion, Pacific J. Math. **174** (1996), no. 2, 431–442.
    - Twisetd Alexander polynomial can be defined as Reidemeister torsion.
- P. Kirk and C. Livingston, Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants, Topology **38**, (1999), no. 3, 635–661.
    - Twisted Alexander polynomial can be defined as the order of twisted Alexander module.
- After 2000, ...lots of many works...

Two survey papers:

- S. Friedl and S. Vidussi, A survey of twisted Alexander polynomials, The mathematics of knots, p45–94, Springer, Heidelberg, 2011.
- T. Morifuji, Representation of knot groups into $SL(2;\mathbb{C})$ and twisted Alexander polynomials, to appear in the book "Handbook of Group Actions (Vol I)", p527–572, Higher Educational Press and International Press, Beijing-Boston.

Twisted Alexander polynomial is a polynomial ?

It is not clear whether a twisted Alexander polynomial is a polynomial or not.

Remark

For any abelian representation $\rho : G(K) \to SL(2;\mathbb{F})$, $\Delta_{K,\rho}(t)$ is not a Laurent polynomial. In this case it can be described by the Alexander polynomial as the one for the trivial representation.

However, under a generic assumption on ρ, the twisted Alexander polynomial is a Laurent polynomial.

Proposition (K.-Morifuji)

If $\rho : G(K) \to SL(2;\mathbb{F})$ is not an abelian representation, then $\Delta_{K,\rho}(t)$ is a Laurent polynomial with coefficients in $\mathbb{F}$.

Example:Figure-eight knot 4_1 again

$$G(4_1) = \langle x, y | wx = yw \rangle \ (w = x^{-1}yxy^{-1})$$

Remark

Here the generators x and y are conjugate by w. This is the point that it is easy to treat $SL(2;\mathbb{C})$-representation for 2-bridge knot.

For simplicity, we write X to $\rho(x)$ for $x \in G(K)$.

Lemma

Let $X, Y \in SL(2,\mathbb{C})$. If X and Y are conjugate and $XY \neq YX$, then there exists $P \in SL(2;\mathbb{C})$ s.t.

$$PXP^{-1} = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, \ PYP^{-1} = \begin{pmatrix} s & 0 \\ u & 1/s \end{pmatrix}.$$

For any irreducible representation ρ, we may assume that its representative of this conjugacy class is given by

$$\rho_{s,u} : G(4_1) \to SL(2;\mathbb{C}) \ (s, u \in \mathbb{C} \setminus \{0\})$$

where

$$X = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, Y = \begin{pmatrix} s & 0 \\ u & 1/s \end{pmatrix}$$

Remark

Because

$$trX = s + \frac{1}{s}, trX^{-1}Y = 2 - u,$$

then the space $\hat{R}$ of the conjugacy classes of the irreducible representations can be parametrized by the traces of $X, X^{-1}Y$.

We compute the matrix

$$R = WX - YW = \rho(w)\rho(x) - \rho(y)\rho(w)$$

to get the defining equations of the space $\hat{R}$.
We compute each entry of $R = (R_{ij})$:

- $R_{11} = 0$,
- $R_{12} = 3 - \frac{1}{s^2} - s^2 - 3u + \frac{u}{s^2} + s^2u + u^2$,
- $R_{21} = -3u + \frac{u}{s^2} + s^2u + 3u^2 - \frac{u^2}{s^2} - s^2u^2 - u^3 = -uR_{12}$,
- $R_{22} = 0$.

Hence $R_{12} = 0$ is the equation of $\hat{R}$ of the conjugacy classes of the irreducible representations.

Proposition

$$\hat{R} = \{(s, u) \in (\mathbb{C} \setminus \{0\})^2 \mid R_{12} = 0\}$$

This equation

$$3 - \frac{1}{s^2} - s^2 - 3u + \frac{u}{s^2} + s^2 u + u^2 = 0$$

can be solved in u:

$$u = \frac{-1 + 3s^2 - s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}.$$

Here we apply $\dfrac{\partial}{\partial y}$ to $wx - yw$;

$$\begin{aligned}\frac{\partial(wx - yw)}{\partial y} &= \frac{\partial w}{\partial y} - 1 - y\frac{\partial w}{\partial y}\\ &= (1-y)\frac{\partial w}{\partial y} - 1\\ &= (1-y)(x^{-1} - wx) - 1.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}A_{\rho,1} &= \Phi\left(\frac{\partial(wx - yw)}{\partial y}\right)\\ &= (E - tY)(t^{-1}X^{-1} - tWX) - E.\end{aligned}$$

We substitute

$$u = \frac{-1 + 3s^2 - s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}$$

to each entry and compute its determinant.
Here we get the following (not depend on the choice of u):

$$\det A_{\rho,1} = \frac{1}{t^2} - \frac{3}{st} - \frac{3s}{t} + 6 + \frac{2}{s^2} + 2s^2 - \frac{3t}{s} - 3st + t^2$$

On the other hand, we obtain

$$\det(tX - E) = t^2 - (s + 1/s)t + 1.$$

Finnaly we obtain

$$\begin{aligned}\Delta_{4_1,\rho_{s,u}}(t) &= \frac{\det A_{\rho,1}}{\det(tX - E)} \\ &= \frac{1}{t^2} - \frac{2\left(1+s^2\right)}{st} + 1 \\ &= \frac{1}{t^2}(t^2 - 2\left(s + \frac{1}{s}\right)t + 1) \\ &= \frac{1}{t^2}(t^2 - 2(trX)t + 1).\end{aligned}$$

Remark

- *Because $\rho_{s,u}$ is not abelian, then $\Delta_{4_1,\rho_{s,u}}(t)$ is a Laurent polynomial.*
- *Because 4_1 is fibered, then $\Delta_{4_1,\rho_{s,u}}(t)$ is monic (explain later).*

Example:torus knots

We consider that $\Delta_{K,\rho}(t)$ is a Laurant polynomial valued function on the space of conjugacy classes of $SL(2;\mathbb{C})$-irreducible representations. We see it has variation on representations. On the other hand, we have the following.

Theorem (K.-Morifuji)

For a torus (p,q)-knot $T(p,q)$, $\Delta_{T(p,q),\rho}(t)$ is a locally constant function on each connected component of the space of $SL(2;\mathbb{C})$-irreducible representations.

- $T(p,q) \subset S^3$ a torus (p,q)-knot
- $G(p,q) = \langle x, y | x^p = y^q \rangle$ its knot group
- $m \in G(p,q)$ the meridian given by $x^{-r}y^s$ where $ps - qr = 1$.
- $z = x^p = y^q$ a center element of the infinite order.
- $\rho : G(p,q) \to SL(2;\mathbb{C})$ an irreducible representation.

Fact

The center of $SL(2;\mathbb{C})$ is $\{\pm E\}$.

Lemma

$$Z = \rho(z) = \pm E.$$

Because $Z = \pm E$, then

$$X^p = \pm E, Y^q = \pm E.$$

Here we may assume the eigenvalues of X and Y are given by

$$\lambda^{\pm 1} = e^{\pm\sqrt{-1}\pi a/p}, \mu^{\pm 1} = e^{\pm\sqrt{-1}\pi b/q},$$

where $0 < a < p, 0 < b < q$. Hence we have

$$X^p = (-E)^a, Y^q = (-E)^b$$

In any case we have

$$X^{2p} = Y^{2q} = E.$$

Now we get

$$trX = 2\cos\frac{\pi a}{p}, trY = 2\cos\frac{\pi b}{q}.$$

Proposition (D. Johnson)

- *The conjugacy class of the irreducible representation ρ is uniquely determined for fixed triple (trX, trY, trM) where*
  - $trX = 2\cos\frac{\pi a}{p}, trY = 2\cos\frac{\pi b}{q}, Z = (-E)^a.$
  - $trM \neq 2\cos\pi(\frac{ra}{p} \pm \frac{sb}{q}).$
  - $0 < a < p$, $0 < b < q$.
  - $a \equiv b$ *mod 2.*

We see each connected component of the conjugacy classes can be parametrized by $tr M$ under fixing (a, b).
Recall $G(p, q) = \langle x, y \mid r = x^p y^{-q} \rangle$. By applying Fox's differentials,

$$\frac{\partial r}{\partial x} = 1 + x + \cdots + x^{p-1}.$$

Then we get

$$\begin{aligned}
&\Delta_{T(p,q),\rho}(t) \\
=&\frac{\Phi(\frac{\partial r}{\partial x})}{\Phi(y-1)} \\
=&\frac{(1 + \lambda t^q + \cdots + \lambda^{p-1} t^{(p-1)q})(1 + \lambda^{-1} t^q + \cdots + \lambda^{-(p-1)} t^{-(p-1)q})}{1 - (\mu + \mu^{-1}) t^p + t^{2p}}
\end{aligned}$$

Torus $(2, q)$-knots

We consider the case of torus $(2, q)$-knot for simplicity. Here the connected components consists of $\frac{q-1}{2}$ components parametrized by odd integer b with $0 < b < q$.

Theorem (K.-Morifuji)

the twisted Alexander polynomial is given by

$$\Delta_{K,\rho_b}(t) = \left(t^2 + 1\right) \prod_{0<k<q,\ k:\mathrm{odd},\ k\neq b} \left(t^2 - \xi_k\right)\left(t^2 - \bar{\xi}_k\right),$$

where $\xi_k = \exp(\sqrt{-1}\pi k/q)$.

Example

In partiquar, for $3_1 = T(2,3)$, there is just one connected component. For any irreducible representation ρ, we have

$$\Delta_{K,\rho}(t) = \frac{t^6+1}{t^4-t^2+1} \\ = t^2+1.$$

Twisted Alexander polynomial = Reidemeister torsion

- $K \subset S^3$ a knot
- $E(K)$ its exterior
- $\alpha : G(K) \cong \pi_1(E(K)) \to \langle t \rangle \subset GL(1; \mathbb{Z}[t, t^{-1}])$
- $\rho : G(K) \cong \pi_1(E(K)) \to SL(2; \mathbb{F})$
- $\rho \otimes \alpha : G(K) \cong \pi_1(E(K)) \to GL(2; \mathbb{F}[t, t^{-1}]) \subset GL(2; \mathbb{F}(t))$
- $C_*(E(K); \mathbb{F}(t)^2_{\rho\otimes\alpha})$ local system by $\rho \otimes \alpha$

We assume

$$H_*(E(K); \mathbb{F}(t)^2_{\rho\otimes\alpha}) = 0$$

Then we can define $\tau_{\rho\otimes\alpha}(E(K)) \in \mathbb{F}(t)$.

Under the acyclicity condition, we have the fillowing.

Theorem

$$\Delta_{K,\rho}(t) = \tau_{\rho\otimes\alpha}(E(K)) \in \mathbb{F}(t).$$

Remark

By using this theorem, we can get some properties of twisted Alexander polynomials.

twisted Alexander polynomial=order ideals

Along the same direction, we have the following.

Theorem (Kirk-Livingston)

- $ord(H_0(E(K); \mathbb{F}[t, t^{-1}]^2_{\rho\otimes\alpha}) = (\det\Phi(x_k - 1))$
- $ord(H_1(E(K); \mathbb{F}[t, t^{-1}]^2_{\rho\otimes\alpha}) = (\det A_{\rho,k}))$

Remark

Sometimes the numerator

$$\det A_{\rho,k} = \textit{a generator of } \operatorname{ord}(H_1(E(K); \mathbb{F}[t, t^{-1}]^2_{\rho\otimes\alpha})$$

is simply called twisted Alxander polynomial.

twisted Alexander polynomial = Obstruction

- $G(K) = \langle x_1, \cdots, x_n | r_1, \cdots, r_{n-1} \rangle$ Wirtinger presentation
- $\rho : G(K) \to SL(2; \mathbb{C})$ a representation
- $X_i = \rho(x_i)$
- $\tilde{X}_i = a \begin{pmatrix} X_i & \mathbf{b}_i \\ \mathbf{0} & 1 \end{pmatrix} \in GL(3; \mathbb{C})$ $(a \in \mathbb{C} \setminus \{0\})$

Problem

When the map $\tilde{\rho}_a : x_i \mapsto \tilde{X}_i$ gives a representation $\tilde{\rho}_a : G(K) \to GL(3; \mathbb{C})$?

We can generalize the theorem by de Rham as folows.

Theorem (Wada, unpublished)

$\tilde{\rho}_a : G(K) \to GL(3;\mathbb{C})$ is a representation if and only if $\Delta_{K,\rho}(a) = 0$.

Hence we can say twisted Alexander polynomial is an obstruction to deform a $SL(2;\mathbb{C})$-representation in $\mathbb{C}^2 \rtimes SL(2;\mathbb{C}) \subset GL(3;\mathbb{C})$.

Part 3:Applications

How to find a linear representation

Twisted Alexander polynomial is an invariant for $G(K)$ with ρ. In general it is not easy to find a linear representation of $G(K)$.
We have two methods to do it by using a computer.

- a linear representation over a finite field
- a finite quotient (e.g, epimorphism onto a symmetric group)

A finite quotient

If we have a finite quotient, which is an epimorphism onto a finite group G:

$$\gamma : G(K) \to G.$$

- G acts neturally on G and $\mathbb{Q}G$.
- $G(K)$ acts on $\mathbb{Q}G$.
- $\dim_{\mathbb{Q}}(\mathbb{Q}G) = |G|$ where $|G|$ is the order of G.

Then this gives a $|G|$-dimensional unimodular representation

$$\tilde{\gamma} : G(K) \to SL(|G|; \mathbb{Q}).$$

Detect the unknot

K is the trivial knot, then

$$\Delta_{K,\mathrm{triv}_l} = \frac{1}{(\lambda_1 t - 1)\cdots(\lambda_l t - 1)}$$

for the trivial l-dimensional representation.
On the other hand, we cann see the following.

Theorem (Silver-Williams)

If K is not trivial, then there exists a finite quotient $\gamma : G(K) \to G$ s.t. $\Delta_{K,\tilde{\gamma}}(t) \neq \frac{1}{(\lambda_1 t-1)\cdots(\lambda_l t-1)}$.

Fibered knot

Definition

K is fibered if $E(K)$ admits a structure of a fiber bundle

$$E(K) = S \times [0,1]/(x,1) \sim (\varphi(x),0)$$

over S^1. Here $\varphi : S \to S$ is a orientation preserving diffeomorphism of a compact oriented surface S.

Theorem (Stallings, Neuwirth)

K is a fibered knot of genus g if and only if $[G(K), G(K)]$ is a free group of rank $2g$.

To check this condition on $[G(K), G(K)]$ is very hard.
The next proposition is well known.

Proposition

$$\Delta_K(t) = \det(t\varphi_* - E : H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z})).$$

Corollary

If K is fibered, then $\Delta_K(t)$ is monic.

In general we define to be monic as follows.

Definition

A Laurent polynomial $f(t)$ over R is monic if its coefficient of the highest degree is a unit in R.

Now we consider twisted Alexander polynomial over a field. Since any non zero element is a unit in a field, then it does not make sense. However for any $SL(n;\mathbb{F})$-representation, twisted Alexander polynomial is well-defined as a rational expression up ± 1.

Definition

$\Delta_{K,\rho}$ is monic if the highest degree coefficients of the denominator and the numerator are ± 1.

Generalization to the twisted case

Theorem (Cha, Goda-Morifuji-K.)

If K is fibered, then $\Delta_{K,\rho}$ is monic for any ρ.

- J. C. Cha, Fibred knots and twisted Alexander invariants, Trans. Amer. Math. Soc. **355** (2003), no. 10, 4187–4200
- H. Goda, T. Kitano and T. Morifuji, Reidemeister torsion, twisted Alexander polynomial and fibered knots, Comment. Math. Helv. **80** (2005), no. 1, 51–61.
    - Using properties of Reidemeister torion

To make refinement we need the notion of Thurston norm.

Thurston norm

Here $\alpha \in H^1(G(K);\mathbb{Z}) = H^1(E(K);\mathbb{Z})$. As

$$H^1(E(K);\mathbb{Z}) \cong H_2(E(K), \partial E(K);\mathbb{Z})$$

by Poincaré duality, there exists an properly embedded surface $S = S_1 \cup \cdots \cup S_k$ whose homology class $[S]$ is dual to α.

Definition

$$\begin{aligned} \chi_-(S) &= \sum_{i=1}^{k} \max\{-\chi(S_i), 0\} \\ &= \sum_{i:\chi(S_i)<0} -\chi(S_i). \end{aligned}$$

Thurston norm $||\alpha||_T$ is defined by the following.

Definition

$$||\alpha||_T = \min\{\chi_-(S) | S \subset E(K) \text{ dual to } \alpha\}$$

Example

If K is a fibered knot of genus g, then its fiber surface S is dual to α.
Here $\chi(S) = 2 - 2g - 1 = 1 - 2g$.
Hence

- $||\alpha||_T = 2g - 1$
- $\deg(\Delta_K(t)) = 2g$.
- $||\alpha||_T = \deg(\Delta_K(t)) - 1 = \deg(\tau_\alpha(E(K))$

This can be generalized for the twisted Alexander polynomial. The next result was turning point.

Theorem (Friedl-Kim)

Let K be a fibered knot with a representation $\rho : G(K) \to SL(l;\mathbb{F})$. Then it holds that

- $\Delta_{K,\rho}(t)$ *is monic*
- $l||\alpha||_T = \deg(\Delta_{K,\rho}(t))$.

The "converse" is true

- S. Friedl and S. Vidussi, Twisted Alexander polynomials detect fibered 3-manifolds, Ann. of Math. (2) **173** (2011), no. 3, 1587–1643.

Theorem (Friedl-Vidussi)

For any representation $\tilde{\gamma}$ induced by a finite quotient $\gamma : G(K) \to G$, if it holds

- $\Delta_{K,\tilde{\gamma}}(t)$ *is monic,*
- $|G| \cdot ||\alpha||_T = \deg(\Delta_{K,\tilde{\gamma}}(t))$,

then K is a fibered knot of genus $g = \frac{\deg(\Delta_{K,\tilde{\gamma}}(t))+|G|}{2|G|}$.

Outline of proof

- $S \subset E(K)$ a Seifert surface s.t. S is dual to α.
- $N(S) = S \times (-1, 1) \subset E(K)$
- $M = E(K) \setminus N(K)$ (sutured manifold)
- From the condition on twisted Alexander polynomials we can see $H_*(M) \cong H_*(S)$ for any twisted coefficient.
- This implies $\pi_1 M \cong \pi_1 S$.
- Hence $M \cong S \times I$.

To detect fiberedness, we have to compute Thurston norm $||\alpha||_T$. In general it is difficult. However we do not need to do.
For a non-fibered knot, we can see the vanishing of a twisted Alexander polynomial.

Theorem (Friedl-Vidussi)

If K is not fibered, then there exists a representation ρ s.t. $\Delta_{K,\rho}(t) = 0$.

For a hyperbolic knot

- N. Dunfield, S. Friedl and N. Jackson, Twisted Alexander polynomials of hyperbolic knots, Exp. Math. **21** (2012), no. 4, 329–352.

- K a hyperbolic knot.
- $\rho_0 : G(K) \to SL(2; \mathbb{C})$ a lift of holonomy representation with $tr(m) = 2$

If K is a fiber knot of genus g, then twisted Alexander polynomial $\Delta_{K,\rho_0}(t)$ is monic polynomial of degree $4g - 2$,

DFJ-conjecture

The converse is true. Namely if $\Delta_{K,\rho_0}(t)$ is monic polynomial of degree $4g - 2$, then K is fibered.

Theorem (DFJ)

DFJ-conjecture is true for any knot with at most 15-crossings.

Theorem (Tran-Morifuji)

DFJ-conjecture is true for any twist knot.

Epimorphism between knot groups

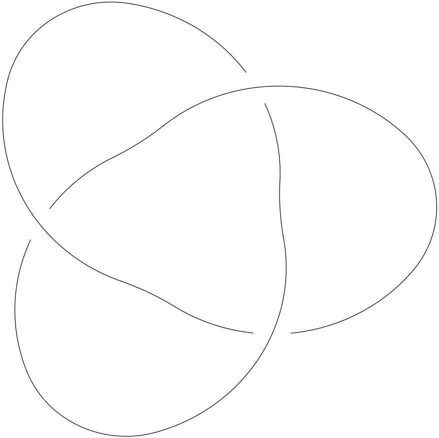
We study epimorphisms, namely surjective homomorphisms, between knot groups.

Definition

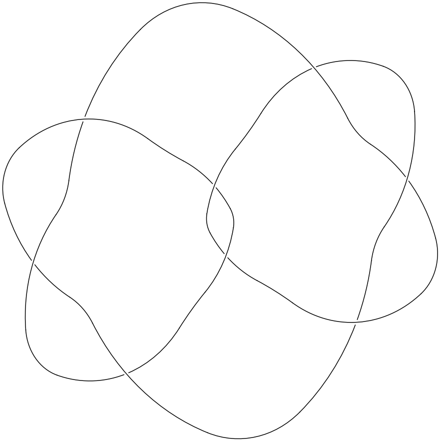
For two knots K_1, K_2, we write $K_1 \geq K_2$ if there exists an epimorphism $\varphi : G(K_1) \to G(K_2)$ which maps a meridian of K_1 to a meridian of K_2.

We start from a simple example $8_5 \geq 3_1$.

3_1



8_5



$8_5 \geq 3_1$

They have the following presentations:

$$
\begin{aligned}
G(8_5) = \langle y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8 \mid & y_7 y_2 y_7^{-1} y_1^{-1}, y_8 y_3 y_8^{-1} y_2^{-1}, \\
& y_6 y_4 y_6^{-1} y_3^{-1}, y_1 y_5 y_1^{-1} y_4^{-1}, \\
& y_3 y_6 y_3^{-1} y_5^{-1}, y_4 y_7 y_4^{-1} y_6^{-1}, \\
& y_2 y_8 y_2^{-1} y_7^{-1} \rangle.
\end{aligned}
$$

$$
G(3_1) = \langle x_1, x_2, x_3 \mid x_3 x_1 x_3^{-1} x_2^{-1}, x_1 x_2 x_1^{-1} x_3^{-1} \rangle.
$$

If generators are mapped to the following words:

$$y_1 \mapsto x_3,\ y_2 \mapsto x_2,\ y_3 \mapsto x_1,\ y_4 \mapsto x_3,$$

$$y_5 \mapsto x_3,\ y_6 \mapsto x_2,\ y_7 \mapsto x_1,\ y_8 \mapsto x_3,$$

any relator in $G(8_5)$ maps to trivial element in $G(3_1)$.

$$y_7 y_2 y_7^{-1} y_1^{-1} \mapsto x_1 x_2 x_1^{-1} x_3^{-1} = 1,$$
$$y_8 y_3 y_8^{-1} y_2^{-1} \mapsto x_3 x_1 x_3^{-1} x_2^{-1} = 1, \cdots$$

Then this gives an epimorphism from $G(8_5)$ onto $G(3_1)$, which maps a meridian to a meridian. Therefore, we can write

$$8_5 \geq 3_1.$$

Geometric meaning

The geometric reason why there exists an epimorphism from $G(8_5)$ to $G(3_1)$ is

- 8_5 has a period 2, namely, it is invariant under some π-rotation of S^3,
- 3_1 is its quotient knot of 8_5.

When and how there exists an epimorphism between knot groups ?
There are some geometric situations as follows.

- To the trivial knot $\bigcirc$:
  For any knot K, then there exists an epimorphism

$$\alpha : G(K) \to G(\bigcirc) = \mathbb{Z}.$$

  This is just the abelianization

$$G(K) \to G(K)/[G(K), G(K)] \cong \mathbb{Z}.$$

  This can be always realized a collapse map with degree 1.

- From any composite knot to each of factor knots:
  There exist two epimorphisms

$$G(K_1 \sharp K_2) \to G(K_1),\ G(K_1 \sharp K_2) \to G(K_2).$$

  They are also just induced by collapse maps with degree 1.
- A degree one map induces an epimorphism: Explain precisely later.
- Periodic knots: Let K be a knot with period n. Its quotient map $(S^3, K) \to (S^3, K') = (S^3, K)/_\sim$ induces an epimorphism

$$G(K) \to G(K')$$

- For any knot K, we take the composite knot $K\sharp\bar{K}$. Then there exist epimorphisms

$$G(K\sharp\bar{K}) \to G(K).$$

This epimorphism is induced from a quotient map

$$(S^3, K\sharp\bar{K}) \to (S^3, K)$$

of a reflection $(S^3, K\sharp\bar{K})$, whose degree is zero.

- Ohtsuki-Riley-Sakuma construction between 2-bridge links : We do not mention precisely here.

A mapping degree

A proper map

$$\varphi : (E(K_1), \partial E(K_1)) \to (E(K_2), \partial E(K_2))$$

induces an homorphism

$$\varphi_* : H_3(E(K_1), \partial E(K_1); \mathbb{Z}) \to H_3(E(K_2), \partial E(K_2); \mathbb{Z}).$$

Definition

A degree of φ is defined to be the integer d satisfying

$$\varphi_*[E(K_1), \partial E(K_1)] = d[E(K_2), \partial E(K_2)]$$

A degree one map induces an epimorphism

Lemma

If $\varphi_* : G(K_1) \to G(K_2)$ is induced from a degree d map, then this degree d can be divisible by the index $n = [G(K_2) : \varphi_*(G(K_1))]$. Namely d/n is an integer.

In particular if $d = 1$, then n should be 1.

Proposition

If there exists a degree one map

$$\varphi : (E(K_1), \partial E(K_1)) \to (E(K_2), \partial E(K_2)),$$

then φ induces an epimorphism

$$\varphi_* : G(K_1) \to G(K_2).$$

Remark

There exists an epimorphism induced from

- *a non zero degree map, but not degree one map,*
- *a degree zero map*

For example, the epimorphism which is induced from a reflection,

$$G(K\sharp\bar{K}) \to G(K)$$

is induced from a degree zero map.

Proposition

The relation $K \geq K'$ gives a partial order on the set of the prime knots. Namely,

1. $K \geq K$
2. $K \geq K',\ K' \geq K \Rightarrow K = K'$
3. $K \geq K',\ K' \geq K" \Rightarrow K \geq K"$

The only one non trivial claim is,

$$K \geq K',\ K' \geq K \Rightarrow K = K'.$$

Here are two key facts to prove it.

- A knot group $G(K)$ is Hopfian, namely any epimorphism $G(K) \to G(K)$ is an isomorphism.
- $G(K)$ determines its knot type of K.

A criterion for the non-existence

- T. Kitano, M. Suzuki and M. Wada, Twisted Alexander polynomials and surjectivity of a group homomorphism, Algebr. Geom. Topol. **5** (2005), 1315–1324.
  - Erratum: Algebr. Geom. Topol. **11** (2011), 2937–2939

Computation for the knot table

- T. Kitano and M. Suzuki, A partial order in the knot table, Experiment. Math. **14** (2005), no. 4, 385–390.
  - Corrigendum to: "A partial order in the knot table". Exp. Math. **20** (2011), no. 3, 371.
  - the knots with up to 10-crossings (=Reidemeister-Rolfsen table)
- K. Horie, T. Kitano, M. Matsumoto and M. Suzuki, A partial order on the set of prime knots with up to 11 crossings, J. Knot Theory Ramifications **20** (2011), no. 2, 275–303.
  - Errata: A partial order on the set of prime knots with up to 11 crossings. J. Knot Theory Ramifications **21** (2012), no. 4, 1292001, 2 pp.
  - the knots with up to 11-crossings.

Fundamental tools to do are

- Alexander polynomial
- Twisted Alexander polynomial
- Computer

By using two invariant, we can prove the non-exisitence of epimorphisms. By using computer, we find it for the rest. The list with up to 11-crossings of the partial order are the following.

Theorem(KS+HKMS)

$$\left.\begin{array}{l} 8_5, 8_{10}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, \\ 9_1, 9_6, 9_{16}, 9_{23}, 9_{24}, 9_{28}, 9_{40}, \\ 10_5, 10_9, 10_{32}, 10_{40}, 10_{61}, 10_{62}, \\ 10_{63}, 10_{64}, 10_{65}, 10_{66}, 10_{76}, 10_{77}, \\ 10_{78}, 10_{82}, 10_{84}, 10_{85}, 10_{87}, 10_{98}, \\ 10_{99}, 10_{103}, 10_{106}, 10_{112}, 10_{114}, \\ 10_{139}, 10_{140}, 10_{141}, 10_{142}, 10_{143}, \\ 10_{144}, 10_{159}, 10_{164} \end{array}\right\} \geq 3_1$$

$$\left.\begin{array}{l}
11a_{43}, 11a_{44}, 11a_{46}, 11a_{47}, 11a_{57}, \\
11a_{58}, 11a_{71}, 11a_{72}, 11a_{73}, 11a_{100}, 11a_{106}, \\
11a_{107}, 11a_{108}, 11a_{109}, 11a_{117}, 11a_{134}, 11a_{139}, \\
11a_{157}, 11a_{165}, 11a_{171}, 11a_{175}, 11a_{176}, \\
11a_{194}, 11a_{196}, 11a_{203}, 11a_{212}, 11a_{216}, 11a_{223}, \\
11a_{231}, 11a_{232}, 11a_{236}, 11a_{244}, 11a_{245}, 11a_{261}, \\
11a_{263}, 11a_{264}, 11a_{286}, 11a_{305}, 11a_{306}, 11a_{318}, \\
11a_{332}, 11a_{338}, 11a_{340}, 11a_{351}, 11a_{352}, 11a_{355}, \\
11n_{71}, 11n_{72}, 11n_{73}, 11n_{74}, 11n_{75}, 11n_{76}, 11n_{77}, \\
11n_{78}, 11n_{81}, 11n_{85}, 11n_{86}, 11n_{87}, 11n_{94}, \\
11n_{104}, 11n_{105}, 11n_{106}, 11n_{107}, 11n_{136}, \\
11n_{164}, 11n_{183}, 11n_{184}, 11n_{185},
\end{array}\right\} \geq 3_1$$

$$\left.\begin{array}{l}
9_{18}, 9_{37}, 9_{40}, 9_{58}, 9_{59}, 9_{60}, \\
10_{122}, 10_{136}, 10_{137}, 10_{138}, \\
11a_{5}, 11a_{6}, 11a_{51}, 11a_{132}, 11a_{239}, 11a_{297}, 11a_{348}, \\
11a_{349}, 11n_{100}, 11n_{148}, 11n_{157}, 11n_{165}
\end{array}\right\} \geq 4_1$$

$$11n_{78}, 11n_{148} \geq 5_1$$

$$10_{74}, 10_{120}, 10_{122}, 11n_{71}, 11n_{185} \geq 5_2$$

$$11a_{352} \geq 6_1$$

$$11a_{351} \geq 6_2$$

$$11a_{47}, 11a_{239} \geq 6_3$$

To decide the partial order relations, how many cases do we have to consider ?

- up to 10 crossings
    - number of knots : 249
    - number of cases : ${}_{249}P_2 = 61,752$
- up to 11 crossings
    - number of knots : $249 + 552 = 801$
    - number of cases : ${}_{801}P_2 =$640,800

Criterions on the existence of epimorphims

The following fact is well known for Alexander polynomial.

Proposition

If $K_1 \geq K_2$, then $\Delta_{K_1}(t)$ can be divisible by $\Delta_{K_2}(t)$.

This can be generalized to the twisted Alxander polynomial as follows.

Theorem (K.-Suzuki-Wada)

If $K_1 \geq K_2$ realized by $\varphi : G(K_1) \to G(K_2)$, then $\Delta_{K_1, \rho_2 \varphi}(t)$ can be divisible by $\Delta_{K_2, \rho_2}(t)$ for any representation $\rho_2 : G(K_2) \to SL(2; \mathbb{F})$.

By using these criterion over a finite prime field, we have checked the non-existence. For the rest, we can find epimorphisms between knot groups by using a computer.

By using the Kawauchi's imitation theory. The next theorem can be proved.

Theorem (Kawauchi)

For any knot K, there exists a knot $\tilde{K}$ such that

- *$\tilde{K}$ is a hyperbolic knot*
- *there exists an epimorphism from $G(\tilde{K})$ onto $G(K)$ induced by a degree one map.*

On the other hand, the following fact is known.

Fact

For any torus knot K, if there exists an epimorphism $\varphi : G(K) \to G(K')$, then K' is also a torus knot.

Now we can consider a Hasse diagram, which is an oriented graph, for this partial ordering as follows.

- a vertex : each prime knot
- an oriented ege : if $K_1 \geq K_2$, then we draw it from the vertex of K_1 to the one of K_2.

More generally the following problem arises.

Problem

How can we understand the structure of this Hasse diagram of the prime knots under this partial order ?

Not so simple!

This Hasse diagram is not so simple as follows.

Proposition

For any two prime knots K_1 and K_2, there exists a prime knot K such that $K \geq K_1$ and $K \geq K_2$.

It can be also done by the imitation theory.

To check the minimality

In our list, we can see that the knots

$$3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3$$

are minimal elements in the set of prime knots with up to 11-crossings.
Here in fact, we can prove that they are minimal in the set of all prime knots.
Now we can see

Theorem (K.-Suzuki)

They are minimal elements in the set of all prime knots.

Open (hard) problem

By our results, the following problem appears naturally.

Problem

If $K_1 \geq K_2$, then the crossing number of K_1 is greater than the one of K_2?

If it is true, it gives another proof of the theorem by Agol and Liu.

Theorem (Agol-Liu)

Any knot group $G(K)$ surjects onto only finitely many knot groups.

Remark

This statement was called the Simon's conjecture.

When and how epimorphisms induced by degree zero maps appear

Boileau-Boyer-Reid-Wang proved the following.

Proposition (BBRW)

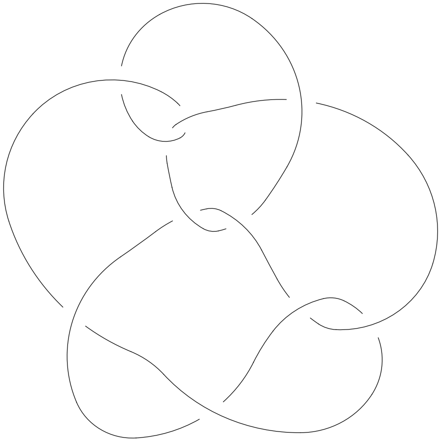
Any epimorphism between 2-bridge hyperbolic knots is always induced from a non zero degree map.

On the other hand, there are some interesting example as follows.

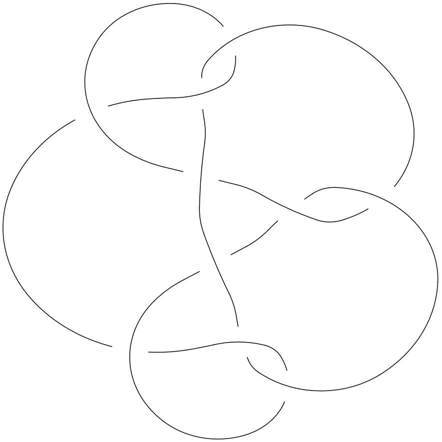
Example

$10_{59},\ 10_{137}$ are 3-bridge hyperbolic knots.

- $10_{59},\ 10_{137} \geq 4_1$.
- There is no non-zero degree map between them. Namely any epimorphism between them is induced from a degree zero map.



10_{59}



10_{137}

Alexander module

To see that there are no non-zero degree maps, we have to study the structure of Alexander modules. The following facts are well known in the theory of surgeries on compact manifolds. For examples, see in the book by Wall.

Fact

If there exits an epimorphism

$$\varphi_* : G(K) \to G(K')$$

induced from a non zero degree map(a degree one map), then its induced epimorphism

$$H_1(\tilde{E}(K); \mathbb{Q}) \to H_1(\tilde{E}(K'); \mathbb{Q})$$

between their Alexander modules is split over $\mathbb{Q}$($\mathbb{Z}$).

Example

We can see the followings by similarly observing Alexander modules.

- $9_{24} \geq 3_1$:any epimorphism between them is induced from an only degree zero map.
- $11a_5 \geq 4_1$:any epimorphism between them is induced from an only degree zero map.

What are these knots?

Remark

Here $10_{59},\ 10_{137}, 9_{24}$ *are Montesinos knots.*

- $10_{59} = M(-1; (5,2), (5,-2), (2,1))$
- $10_{137} = M(0; (5,2), (5,-2), (2,1))$
- $9_{24} = M(-1; (3,1), (3,2), (2,1))$

Construction by ORS

How there exists an epimorphism between them ?
Recall the geometric observation by Ohtsuki-Riley-Sakuma.
Here we assume that

$$\varphi : G(K) \to G(K')$$

is an epimorphism.
We take a simple closed curve $\gamma \subset S^3 \cup K$ which belongs to $\mathrm{Ker}\varphi \subset G(K)$. Then if γ is an unknot in S^3, by taking the surgery along γ, we get a new knot $\tilde{K}$ in S^3 such that there exists an epimorphism $G(\tilde{K}) \to G(K')$.

Apply to $4_1 \sharp \bar{4}_1$

We can apply this construction to $4_1 \sharp \bar{4}_1 = 4_1 \sharp 4_1$. First we recall that there exists an epimorphism

$$G(4_1 \sharp \bar{4}_1) \to G(4_1)$$

which is a quotient map of a reflection. Then it is induced from a degree zero map. By surgery along some simple closed curve, we get both of

$$G(10_{59}) \to G(4_1),$$

and

$$G(10_{137}) \to G(4_1),$$

More generally we can see the following. It was not written explicitely, but essentially in the paper by Ohtsuki-Riley-Sakuma.

Proposition

For any 2-bridge knot K, there exists a Montesinos knot $\tilde{K}$ such that there exists an epimorphism

$$G(\tilde{K}) \to G(K)$$

induced from a degree zero map $E(\tilde{K}) \to E(K)$.

Return to the list of knots with up to 10-crossings. We can find epimorphisms explicitely, but not find all epimorphisms if there exist. For the epimorpshism we could find, the following partial order relations can be realized by epimorphisms induced from degree zero maps.

$$\left.\begin{array}{l} 8_{10},\ 8_{20},\ 9_{24},\ 10_{62},\ 10_{65},\ 10_{77}, \\ 10_{82},\ 10_{87},\ 10_{99},\ 10_{140},\ 10_{143} \end{array}\right\} \geq 3_1$$

$$10_{59},\ 10_{137} \geq 4_1$$

In this list, Montesinos knots appear as follows.

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$$10_{59},\ 10_{157} \geq 4_1$$

In this list, Montesinos knots appear as above.

Remark

The other knots are given by Conway's notation as follows:

- $10_{82} = 6 * *4.2$,
- $10_{87} = 6 * *22.20$,
- $10_{99} = 6 * *2.2.20.20$

As another application of Kawauchi's theory, we can see the following.

Proposition

For any knot K, there exists a hyperbolic knot K' s.t. there exist two epimorphisms from $G(K')$ onto $G(K)$. Further the one is induced by degree one map and another one induced by degree zero map.

Problems

Problem

- *Characterize a minimal knot in the set of prime knots under the partial order.*
- *Characterize an epimorphism induced from a degree zero map.*
- *If* $K_1 \geq K_2$ *then* $\mathrm{vol}(K_1) \geq \mathrm{vol}(K_2)$ *?*
- *How strong is twisted Alexander polynomial for a represenation over a finite field ?*
    - *To determine the non-existence of an epimorphism*
    - *To detect the fiberedness*
- *Find "skein relation " for twisted Alexander polynomial.*

L^2-invariant

How can we treat for an infinite dimensional representation of $G(K)$?

Example

$G(K)$ acts on $\mathbb{C}G(K)$. $\alpha_\theta : G(K) \ni \mapsto \theta \in \mathbb{C}$. Then we can consider

- $A = \left(\theta \dfrac{\partial r_i}{\partial x_j}\right)$
- $A_k \in M((n-1) \times (n-1); \mathbb{C}G(K))$

Here we cannot apply usual $\det$ to $A_k \in M((n-1) \times (n-1); \mathbb{C}G(K))$ since $\mathbb{C}G(K)$ is not commutative.
The problem is

- how to define

$$\text{“}\mathrm{det}\text{''} : M((n-1) \times (n-1); \mathbb{C}G(K)) \to \mathbb{C}.$$

Fundamental idea

- For $A \in GL((n-1) \times (n-1); \mathbb{C})$

$$\log |\mathrm{det} A| = \frac{1}{2} \log \mathrm{A}^* A$$

- For $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ where $a_1, a_2 > 0$,

$$\begin{aligned} \log \det(A) &= \log(a_1 a_2) \\ &= \log(a_1) + \log(a_2) \\ &= tr(\log(A)) \end{aligned}$$

Then the determinant can be defined by trace and log.

Formally we can define the log for a matrix over $\mathbb{C}G(K)$ as a infinite sequense of a matrix. The problem is how to define a good trace.

- $$\mathbb{C}G(K) \ni \sum n_g g \mapsto n_1 \in \mathbb{C}.$$
- For $A \in GL(n-1; \mathbb{C}G(K))$ the trace can be defined by taking the above map for the sum of the diagonals.

This determinant is called Fuglede-Kadison determinant.

Remark

To use the functional analysis we take the l^2-completion $l^2(G(K))$ of $\mathbb{C}G(K)$.

By using this Fuglede-Kadison determinant, we define

- L^2-Alexander polynomial (L^2-Alexander invariant)
- L^2-torsion

There are many and long ways...