Ordered groups, knots, braids and hyperbolic 3-manifolds
Minicourse in Caen

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Outline

Lecture 1: Introduction to ordered groups
Lecture 2: Ordering knot groups; Fibred knots and surgery
Lecture 3: Braids, $Aut(F_n)$ and minimal volume hyperbolic 3-manifolds

Ordered groups

A group is left-ordered if there is a strict total ordering $<$ of its elements such that $g < h$ implies $fg < fh$. Left-orderable groups are also right-orderable, but by a possibly different ordering.
If a group has a strict total ordering $<$ which is both right- and left-invariant, we call it bi-ordered.

Examples

- $\mathbb{Z}^n$ is bi-orderable.
- Free groups are bi-orderable.
- Braid groups are left-orderable (Dehornoy) but not bi-orderable
- Pure braid groups are bi-orderable.
- Surface groups are bi-orderable, except the Klein bottle group

$$\langle a, b | \ a^2 = b^2 \rangle$$

which is only left-orderable, and the projective plane's group $\mathbb{Z}/2\mathbb{Z}$ which is not even left-orderable.

Examples

- $Homeo^+(\mathbb{R})$ is left-orderable.

This can be seen by well-ordering $\mathbb{Q} = \{x_1, x_2, \dots\}$ and comparing functions $f, g : \mathbb{R} \to \mathbb{R}$ by declaring $f \prec g$ iff $f(x_i) < g(x_i)$ at the first i at which $f(x_i)$ and $g(x_i)$ differ.
Moreover, if G is a countable left-orderable group, then G is isomorphic with a subgroup of $Homeo^+(\mathbb{R})$.

Properties of orderable groups

- Left-ordered groups G are torsion-free, that is there are no elements of finite order.
  To see this, suppose $g \in G$ and $g \neq 1$. If $g > 1$, then $g^2 > g > 1$, etc. – all powers of g are greater than 1. Similarly, if $g < 1$, no power of g can be the identity.
- Bi-ordered groups have unique roots: $g^n = h^n, \ n > 0 \implies g = h$
  In a bi-ordered group, one can multiply inequalities: $g < h$ and $g' < g'$ imply $gg' < hh'$. So if $g < h$, we conclude $g^2 < h^2$, then $g^3 < h^3$, etc. That is if g and h are unequal, then their powers g^n and h^n are also unequal.

Properties of orderable groups

- In a bi-ordered group, if g commutes with $h^n, n \neq 0$, then g commutes with h.
  Exercise 1: Prove this (hint: compare g with $h^{-1}gh$).

Properties of orderable groups

Recall that the group ring RG of a group G, with coefficients in a ring R, consists of formal linear combinations of group elements with R coefficients. A typical element is of the form

$$\sum_{i=1}^{m} r_i g_i \quad r_i \in R, g_i \in G.$$

Multiplication is defined as for polynomials:

$$(\sum_{i=1}^{m} r_i g_i)(\sum_{j=1}^{n} s_j h_j) = \sum_{i,j} r_i s_j g_i h_j$$

Properties of orderable groups

Note that if a group G has a torsion element, say $g \in G$ has order 5, then we have an equation:

$$(1 + g + g^2 + g^3 + g^4)(1 - g) = 1 - g^5 = 0.$$

The two terms on the left are nonzero in $\mathbb{Z}G$, yet their product equals zero. Such elements are called zero divisors.

Our example illustrates that if G contains elements of finite order, then $\mathbb{Z}G$ has zero divisors.

Conjecture: If the ring R has no zero divisors and G is torsion-free, then RG has no zero divisors.

This is unsolved, even for the case $R = \mathbb{Z}$

Properties of orderable groups

- Left-orderable groups satisfy the zero-divisor conjecture, that is, if R has no zero divisors and G is left-orderable, then RG has no zero divisors.
  Proof: Consider a product $(\sum_{i=1}^{m} r_i g_i)(\sum_{j=1}^{n} s_j h_j) = \sum_{i,j} r_i s_j g_i h_j$, where we assume that the r_i and s_j are all nonzero, the g_i are distinct and the h_j are written in strictly ascending order, with respect to a given left-ordering of G.
  At least one of the group elements $g_i h_j$ on the right-hand side is minimal in the left-ordering. If $j > 1$ we have, by left-invariance, that $g_i h_1 < g_i h_j$ and $g_i h_j$ is not minimal. Therefore we must have $j = 1$. On the other hand, since we are in a group and the g_i are distinct, we have that $g_i h_1 \neq g_k h_1$ for any $k \neq i$. We have established that there is exactly one minimal term on the r.h.s. It follows that it survives any cancellation, and so the r.h.s. cannot be zero (because $r_i s_1 \neq 0$). Thus RG has no zero divisors.

Properties of orderable groups

Exercise 2: Show that if R and G are as above, then the only units (invertible elements) of RG are "monomials" of the form rg, where r is an invertible element of R.

- (LaGrange, Rhemtulla) If G is left-orderable and H is any group, then $\mathbb{Z}G \cong \mathbb{Z}H \implies G \cong H$.
- If G is bi-ordered, then $\mathbb{Z}G$ embeds in a division ring.

Properties of orderable groups

If $(G, <)$ is a left-ordered group, then the positive cone

$$P = P_< = \{g \in G | 1 < g\}$$

is a semigroup ($P \cdot P \subset P$) and G is partitioned as

$$G = P \sqcup P^{-1} \sqcup \{1\}$$

.

Conversely, if a group G has a sub-semigroup with the above properties, then G can be left-ordered by the rule

$$g < h \Leftrightarrow g^{-1}h \in P$$

Exercise 3: Verify that this recipe defines a left order of G.
The left-ordering is a bi-ordering iff its positive cone is normal:

$$g^{-1}Pg \subset P \quad \forall g \in G$$

Topology on the power set of a set

We begin with a reminder of the Tychonoff topology of a cartesian product of spaces. It is the smallest topology such that the projection functions are continuous.

If X is any set, the power set $\mathcal{P}(X)$ can be identified with the set $2^X = \{0,1\}^X$ of all functions $f : X \to \{0,1\}$ via the correspondence of subsets with their characteristic functions:

$$Y \subset X \leftrightarrow f_Y$$

where $f_Y(x) = 1 \iff x \in Y$.

Giving $\{0,1\}$ the discrete topology, $\{0,1\}^X$ is a special case of a product space and can be given the Tychonoff topology.

Topology on the power set of a set

This then defines a topology on $\mathcal{P}(X) \cong \{0,1\}^X$.
Typical open sets in $\mathcal{P}(X)$ are
$U_x = \{Y \subset X | x \in Y\} \cong \{f : X \to \{0,1\} | f(x) = 1\}$
and $U_x^c = \{Y \subset X | x \notin Y\}$.
Finite intersections of such sets form a basis for the "Tychonoff" topology of $\mathcal{P}(X)$
By a theorem of Tychonoff, it is compact.
It is also totally disconnected: if Y_1 and Y_2 are distinct elements of $\mathcal{P}(X)$, choose an x which is in Y_1 (say) but not in Y_2. Then the sets U_x and U_x^c form a separation of $\mathcal{P}(X)$ with $Y_1 \in U_x$ and $Y_2 \in U_x^c$.
If X is countably infinite, $\mathcal{P}(X)$ is homeomorphic with the Cantor set.

Topology on the set of orderings

The set $LO(G)$ of left-orderings $<$ of a group G can therefore be identified with the set of subsets $P \subset G$ (i. e. elements of $\mathcal{P}(G)$) satisfying
(1) $P \cdot P \subset P$ and
(2) $G = P \sqcup P^{-1} \sqcup \{1\}$

Proposition

$LO(G)$ is a closed subset of $\mathcal{P}(G)$.

Proof: The set of $P \subset G$ which do not satisfy (1) is exactly the union over all $g, h \in G$ of the sets $U_g \cap U_h \cap U_{gh}^c$, and is therefore open. Similarly, one checks that (2) is a closed condition.

Topology on the set of orderings

Corollary

$LO(G)$ is compact and totally disconnected.

Recall that we have been identifying left-orderings with their positive cones. A basic neighborhood of a left-ordering $<$ of a group G can be defined by considering a finite number of inequalities $g_i < h_i$ – the neighborhood of $<$ is all orderings $\prec$ for which those inequalities remain true: $g_i \prec h_i$.
One can also check that the set of all bi-orders of a group G is closed in $LO(G)$, hence also forms a compact, totally disconnected space.... Possibly empty!

Criteria for orderability

A basic question regarding a group G is whether it is left-orderable, or in other words, is $LO(G)$ nonempty?

If G is nontrivial, a necessary condition for left-orderability is that G be torsion-free. But that is by no means sufficient.

Suppose G is finitely-generated with generating set S, and let $B_n(G)$ be the n-ball in the Cayley graph of G with respect to the set S. This is the set of all elements of G which can be written as a product of n or fewer elements of S and their inverses.

Criteria for orderability

Call a subset Q of $B_n(G)$ a pre-order if
(1') $(Q \cdot Q) \cap B_n(G) \subset Q$ and
(2') $B_n(G) = Q \sqcup Q^{-1} \sqcup \{1\}$

Proposition

If G is left-orderable, then every $B_n(G)$ has a pre-order.

This is the basis for a finite algorithm to test for orderability of a f. g. group: see N. Dunfield's website.
Perhaps surprisingly, the converse also is true.

Criteria for orderability

Theorem

If every $B_n(G)$ has a pre-order, then G is left-orderable.

Proof: Note that the restriction to $B_n(G)$ of a pre-order for $B_{n+1}(G)$ is a pre-order for $B_n(G)$.
Let $\mathcal{Q}_n = \{R \subset G | R \cap B_n(G)$ is a pre-order for $B_n(G)\}$.
One checks that $\mathcal{Q}_n$ is closed in $\mathcal{P}(G)$. In a compact space, the intersection of a nested sequence of nonempty closed sets is nonempty. It is easy to check that if a set $P \subset G$ is in every $B_n(G)$, then P satisfies (1) and (2).

Thus we have $LO(G) = \bigcap_{n=1}^{\infty} \mathcal{Q}_n \neq \emptyset$.

Criteria for orderability

The assumption of being finitely-generated is not really essential.

Theorem

A group is left-orderable if and only each of its finitely-generated subgroups is left-orderable.

Proof: The forward implication is obvious. For the reverse implication, consider any finite subset F of the given group G and let $\langle F \rangle$ denote the subgroup of G generated by F. Define

$$\mathcal{Q}(F) := \{Q \subset G | Q \cap \langle F \rangle \text{ is a positive cone for } \langle F \rangle\}$$

For each finite $F \subset G$, $\mathcal{Q}(F)$ is a closed nonempty subset of $\mathcal{P}(G)$.

Criteria for orderability

The family of all $\mathcal{Q}(F)$, for finite $F \subset G$, is a collection of closed sets which has the finite intersection property, because

$$\mathcal{Q}(F_1 \cup F_2 \cup \cdots \cup F_n) \subset \mathcal{Q}(F_1) \cap \mathcal{Q}(F_2) \cap \cdots \cap \mathcal{Q}(F_n).$$

By compactness, the entire family must have a nonempty intersection:

$$LO(G) = \bigcap_{F \subset G \text{finite}} \mathcal{Q}(F) \neq \emptyset.$$

Criteria for orderability

Corollary

An abelian group is bi-orderable if and only if it is torsion-free.

Proof: Bi-orderable groups are torsion-free. To see the other direction, it is enough to observe that a finitely-generated torsion-free abelian group is isomorphic with $\mathbb{Z}^n$ for some finite n. □

Criteria for orderability

Theorem

A group G can be left-ordered if and only if for every finite subset $\{x_1, \ldots, x_n\}$ of $G \setminus \{1\}$, there exist $\epsilon_i = \pm 1$ such that $1 \notin S(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$.

One direction is clear, for if $<$ is a left-ordering of G, just choose ϵ_i so that $x_i^{\epsilon_1}$ is greater than the identity. For the converse, we may assume that G is finitely generated, and we need only show that each k-ball $B_k(G)$, with respect to a fixed finite generating set, has a pre-order. Now consider $\{x_1, \ldots, x_n\}$ to be the entire set $B_k(G) \setminus \{1\}$, and choose $\epsilon_i = \pm 1$ such that $1 \notin S(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$.
We can check that the set $\{x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}\}$ is a pre-order of $B_k(G)$, completing the proof.

Criteria for orderability

Theorem (Burns-Hale)

A group G is left-orderable if and only if for every finitely-generated subgroup $H \neq \{1\}$ of G, there exists a left-orderable group L and a nontrivial homomorphism $H \to L$.

Proof: One direction is obvious. To prove the other direction, assume the subgroup condition. The result will follow if one can show:
Claim: For every finite subset $\{x_1, \ldots, x_n\}$ of $G \setminus \{1\}$, there exist $\epsilon_i = \pm 1$ such that $1 \notin S(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$.
We will establish this claim by induction on n.

Criteria for orderability

It is certainly true for $n = 1$, for $S(x_1)$ cannot contain the identity unles x_1 has finite order, which is impossible since the cyclic subgroup $\langle x_1 \rangle$ must map nontrivially to a left-orderable (hence torsion-free) group.
Next assume the claim true for all finite subsets of $G \setminus \{1\}$ having fewer than n elements, and consider $\{x_1, \ldots, x_n\} \subset G \setminus \{1\}$.
By hypothesis, there is a nontrivial homomorphism

$$h : \langle x_1, \ldots, x_n \rangle \to L$$

where $(L, \prec)$ is a left-ordered group. Not all the x_i are in the kernel; we may assume they are numbered so that

$$h(x_i) \begin{cases} \neq 1 \text{ if } i = 1, \ldots, r, \\ = 1 \text{ if } r < i \leq n. \end{cases}$$

Criteria for orderability

Now choose $\epsilon_1, \dots, \epsilon_r$ so that $1 \prec h(x_i^{\epsilon_i})$ in L for $i = 1, \dots, r$.
For $i > r$, the induction hypothesis allows us to choose $\epsilon_i = \pm 1$ so that $1 \notin S(x_{r+1}^{\epsilon_{r+1}}, \dots, x_n^{\epsilon_n})$.
We now check that $1 \notin S(x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n})$ by contradiction. Suppose that 1 *is* a product of some of the $x_i^{\epsilon_i}$. If all the i are greater than r, this is impossible, as $1 \notin S(x_{r+1}^{\epsilon_{r+1}}, \dots, x_n^{\epsilon_n})$. On the other hand if some i is less than or equal to r, we see that h must send the product to an element strictly greater than the identity in L, again a contradiction. □

Criteria for orderability

A group is indicable if there is a surjection of the group to $\mathbb{Z}$, the infinite cyclic group.

A group is locally indicable if every nontrivial finitely generated subgroup is indicable.

Corollary

If a group is locally indicable, then it is left-orderable.

Criteria for orderability

Exercise 4: Verify that the properties of being torsion-free, left-orderable or locally indicable are preserved under extensions. That is, if $1 \to K \to G \to H \to 1$ is exact and K and H have the given property, then so does G.

This is not the case for bi-orderability. The Klein bottle group demonstrates this.

Example: Let $G = \langle x, y | x^{-1}yx = y^{-1} \rangle$ be the Klein bottle group (fundamental group of the Klein bottle). Let K be the subgroup generated by y.

Exercise 5: Verify that K is normal in G and isomorphic to $\mathbb{Z}$, the group of integers. Moreover $H := G/K$ is also isomorphic to $\mathbb{Z}$.

Therefore we have an exact sequence $1 \to \mathbb{Z} \to G \to \mathbb{Z} \to 1$ and can conclude that G is left-orderable, and in fact locally indicable. Yet it is not bi-orderable, because if it were, the defining relation would imply the contradiction that y is positive if and only if y^{-1} is positive.

Bon Anniversaire, Patrick !