Ordered groups, knots, braids and hyperbolic 3-manifolds
Minicourse in Caen

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Outline

Lecture 1: Introduction to ordered groups

Lecture 2: Ordering knot groups; Fibred knots and surgery

Lecture 3: Braids, $Aut(F_n)$ and minimal volume hyperbolic 3-manifolds

Knot groups

Knot groups and their orderability.

Recall that we discussed orderability of groups and the closely related concept of local indicability.

We have the following implications among these properties:

Bi-orderable $\Longrightarrow$ Locally indicable $\Longrightarrow$ Left-orderable $\Longrightarrow$ Torsion-free

None of these implications is reversible.

Knot groups

If K is a knot in $\mathbb{S}^3$, its knot group is $\pi_1(\mathbb{S}^3 \setminus K)$.
Our goal is to show that all knot groups are left-orderable, in fact locally indicable.
This will be a special case of a more general result about 3-dimensional manifolds.

Knot groups

We will need a few ideas from 3-manifold theory.
Definition: A 3-manifold is *irreducible* if every tame 2-sphere in the manifold bounds a 3-dimensional ball in the manifold.
A nontrivial fact is that if $\tilde{X} \to X$ is a covering space, with X (and therefore $\tilde{X}$) a 3-manifold, then X is irreducible if and only if $\tilde{X}$ is irreducible.
If $X = \mathbb{S}^3 \setminus K$ is a knot complement, then X is irreducible. This is also true if K is a link if (and only if) it is not a split link.
By Alexander duality, we also have that $H_1(\mathbb{S}^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$. That is, the first Betti number (the number of copies of $\mathbb{Z}$ in the first homology group) equals one.

Knot groups

Theorem

Suppose X is a connected, orientable, irreducible 3-manifold (possibly with boundary). If X has positive first Betti number, then $\pi_1(X)$ is locally indicable, and therefore left-orderable.

The proof, essentially due to Howie and Short, will be given below.

Corollary

Knot groups are locally indicable.

Knot groups are locally indicable

Consider X as in the hypothesis of the theorem.
$\pi_1(X)$ is indicable, using the (surjective) Hurewicz homomorphism and a further homomorphism to one of the $\mathbb{Z}$ factors of $H_1(X)$.

$$\pi_1(X) \to H_1(X) \to \mathbb{Z}$$

To show $\pi_1(X)$ is locally indicable, consider a finitely generated nontrivial subgroup $H < \pi_1(X)$. We need to find a surjection $H \to \mathbb{Z}$.

Case 1: H has finite index. This is easy; the Hurewicz map takes H to a finite index subgroup of $H_1(X)$, which therefore contains a copy of $\mathbb{Z}$.

Knot groups are locally indicable

Case 2: H has infinite index. Then there is a covering $p : \tilde{X} \to X$ with $p_*\pi_1(\tilde{X}) = H$. $\tilde{X}$ is noncompact, but its fundamental group is f. g. so, by a theorem of Scott, there is a compact submanifold $C \subset \tilde{X}$ with inclusion inducing an isomorphism $\pi_1(C) \cong \pi_1(\tilde{X}) \cong H$.

C necessarily has nonempty boundary. If $B \subset \partial C$ is a boundary component which is a sphere, then irreducibility implies that B bounds a 3-ball in $\tilde{X}$. That 3-ball either contains C or its interior is disjoint from C, and the former can't happen because that would imply the inclusion map $\pi_1(C) \to \pi_1(\tilde{X})$ is trivial. Therefore, we can adjoin that 3-ball to C removing B as a boundary component and not changing $\pi_1(C)$.

Knot groups are locally indicable

This process allows us to assume that ∂C is nonempty and has infinite homology groups.

Exercise 6: Conclude that C also has infinite homology. [Hint: one way to do this is by considering the Euler characteristic of the closed 3-manifold 2C, obtained by glueing two copies of C together along the boundary.]

Then we have surjections $H \cong \pi_1(C) \to H_1(C) \to \mathbb{Z}$ as required. □

Fibred knots

It is well known that every (tame) knot in $\mathbb{S}^3$ is the boundary of a compact orientable surface (called a Seifert surface) in $\mathbb{S}^3$.
A knot is said to be fibred if there is a fibre bundle map $\mathbb{S}^3 \setminus K \to \mathbb{S}^1$ with fibres being open orientable surfaces whose closures have K as boundary in $\mathbb{S}^3$.
In other words, the complement of K in $\mathbb{S}^3$ can be filled with a circle's worth of orientable surfaces.

Fibred knots

If K is a fibred knot, with complement $X = \mathbb{S}^3 \setminus K$ and with fibre F an open surface, the exact homotopy sequence of a fibration gives the short exact sequence:

$$1 \to \pi_1(F) \to \pi_1(X) \to \pi_1(S^1) \to 1.$$

But $\pi_1(F)$ is a free group and $\pi_1(S^1) \cong \mathbb{Z}$. Both these groups are locally indicable, so we conclude from Exercise 4 that the knot group $\pi_1(X)$ is locally indicable, and therefore left orderable.
That is, the group of a fibred knot is seen to be locally indicable without the need for the general theorem we have proved, which applies to all knots.

Fibred knots

A fibration $X \to S^1$ with fibre F can be considered as the mapping cylinder of a (monodromy) homeomorphism $h : F \to F$:

$$X \cong \frac{F \times [0,1]}{(x,1) \sim (h(x),0)}$$

For a fibred knot with $X = \mathbb{S}^3 \setminus K$ the Alexander polynomial is just the characteristic polynomial of the homology monodromy $H_1(F) \to H_1(F)$. Non-fibred knots also have an Alexander polynomial, but it may not be monic, as is the case for fibred knots.

Fibred knots

Also, the knot group $\pi_1(X)$ is an HNN extension of the free group $\pi_1(F)$, corresponding to the homotopy monodromy $h_* : \pi_1(F) \to \pi_1(F)$, where $\pi_1(F) \cong \langle x_1, \ldots, x_{2g} \rangle$ is a free group.

$$\pi_1(X) \cong \langle x_1, \ldots, x_{2g}, t | h_*(x_i) = t x_i t^{-1} \rangle$$

Exercise 7: This group is bi-orderable if and only if there is a bi-ordering of $\pi_1(F)$ which is preserved by h_*.

Fibred knots

We will sketch the proofs of two theorems regarding bi-ordering fibred knot groups.

Theorem

1. *(Perron - R.) If K is fibred and $\Delta_K(t)$ has all roots real and positive, then its group is bi-orderable.*
2. *(Clay-R.) If K is fibred and its group is bi-orderable, then $\Delta_K(t)$ has some real positive roots.*

Before proving these theorems, we consider some examples.

Examples

Torus knots: curves which can be inscribed on the surface of an unknotted torus in $\mathbb{S}^3$. For relatively prime integers p, q the torus knot $T_{p,q}$ has group

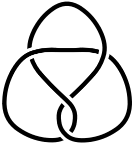
$$\langle a, b | a^p = b^q \rangle.$$

Note that a commutes with b^q but not with b (unless the group is abelian, and the knot unknotted). We've already observed that in a bi-orderable group, if an element commutes with a nonzero power of another element, then the elements must themselves commute. Therefore:

Proposition

The group of a nontrivial torus knot is not bi-orderable.

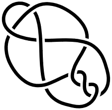
Examples

The figure-eight knot 4_1  is fibred and has Alexander polynomial $\Delta_{4_1} = t^2 - 3t + 1$ with roots $\dfrac{3 \pm \sqrt{5}}{2}$, both real and positive. From Theorem 2 we conclude

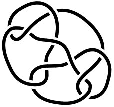
Proposition

The group of the knot 4_1 is bi-orderable.

More bi-orderable knot groups

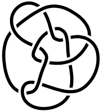


8_{12} $\Delta = 1 - 7t + 13t^2 - 7t^3 + t^4$



10_{137} $\Delta = 1 - 6t + 11t^2 - 6t^3 + t^4$

$11a_5$ $\Delta = 1 - 9t + 30t^2 - 45t^3 + 30t^4 - 9t^5 + t^6$

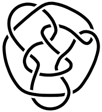


$11n_{142}$ $\Delta = 1 - 8t + 15t^2 - 8t^3 + t^4$

More bi-orderable knot groups

$12a_{0125}$ $\Delta = 1 - 12t + 44t^2 - 67t^3 + 44t^4 - 12t^5 + t^6$

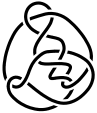
$12a_{0181}$ $\Delta = 1 - 11t + 40t^2 - 61t^3 + 40t^4 - 11t^5 + t^6$



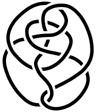
$12a_{1124}$ $\Delta = 1 - 13t + 50t^2 - 77t^3 + 50t^4 - 13t^5 + t^6$

$12n_{0013}$ $\Delta = 1 - 7t + 13t^2 - 7t^3 + t^4$

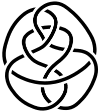
More bi-orderable knot groups



$12n_{0145}$ $\Delta = 1 - 6t + 11t^2 - 6t^3 + t^4$



$12n_{0462}$ $\Delta = 1 - 6t + 11t^2 - 6t^3 + t^4$



$12n_{0838}$ $\Delta = 1 - 6t + 11t^2 - 6t^3 + t^4$

More non bi-orderable knot groups

Recall the Theorem: fibred and bi-orderable $\Longrightarrow$ Δ has positive roots.
This can be used for an alternative proof that torus knots $T_{p,q}$, which are fibred, have non-bi-orderable group, because

$$\Delta_{T(p,q)} = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$$

whose roots are on the unit circle and not real.
There are many other fibred knots which have non-biorderable group for similar reasons ....

More non bi-orderable knot groups

The prime knots with 12 or fewer crossings which are known to have non-bi-orderable group, because they are fibred and have Alexander polynomials without positive real roots, are as follows:

3_1, 5_1, 6_3, 7_1, 7_7, 8_7, 8_{10}, 8_{16}, 8_{19}, 8_{20}, 9_1, 9_{17}, 9_{22}, 9_{26}, 9_{28}, 9_{29}, 9_{31}, 9_{32}, 9_{44}, 9_{47}, 10_5, 10_{17}, 10_{44}, 10_{47}, 10_{48}, 10_{62}, 10_{69}, 10_{73}, 10_{79}, 10_{85}, 10_{89}, 10_{91}, 10_{99}, 10_{100}, 10_{104}, 10_{109}, 10_{118}, 10_{124}, 10_{125}, 10_{126}, 10_{132}, 10_{139}, 10_{140}, 10_{143}, 10_{145}, 10_{148}, 10_{151}, 10_{152}, 10_{153}, 10_{154}, 10_{156}, 10_{159}, 10_{161}, 10_{163}, $11a_9$, $11a_{14}$, $11a_{22}$, $11a_{24}$, $11a_{26}$, $11a_{35}$, $11a_{40}$, $11a_{44}$, $11a_{47}$, $11a_{53}$, $11a_{72}$, $11a_{73}$, $11a_{74}$, $11a_{76}$, $11a_{80}$, $11a_{83}$, $11a_{88}$, $11a_{106}$, $11a_{109}$, $11a_{113}$, $11a_{121}$, $11a_{126}$, $11a_{127}$, $11a_{129}$, $11a_{160}$, $11a_{170}$, $11a_{175}$, $11a_{177}$, $11a_{179}$, $11a_{180}$, $11a_{182}$, $11a_{189}$, $11a_{194}$, $11a_{215}$, $11a_{233}$, $11a_{250}$, $11a_{251}$, $11a_{253}$, $11a_{257}$, $11a_{261}$, $11a_{266}$, $11a_{274}$, $11a_{287}$, $11a_{288}$, $11a_{289}$, $11a_{293}$, $11a_{300}$, $11a_{302}$, $11a_{306}$, $11a_{315}$, $11a_{316}$,

More non bi-orderable knot groups

$11a_{326}$, $11a_{330}$, $11a_{332}$, $11a_{346}$, $11a_{367}$, $11n_{7}$, $11n_{11}$, $11n_{12}$, $11n_{15}$, $11n_{22}$, $11n_{23}$, $11n_{24}$, $11n_{25}$, $11n_{28}$, $11n_{41}$, $11n_{47}$, $11n_{52}$, $11n_{54}$, $11n_{56}$, $11n_{58}$, $11n_{61}$, $11n_{74}$, $11n_{76}$, $11n_{77}$, $11n_{78}$, $11n_{82}$, $11n_{87}$, $11n_{92}$, $11n_{96}$, $11n_{106}$, $11n_{107}$, $11n_{112}$, $11n_{124}$, $11n_{125}$, $11n_{127}$, $11n_{128}$, $11n_{129}$, $11n_{131}$, $11n_{133}$, $11n_{145}$, $11n_{146}$, $11n_{147}$, $11n_{149}$, $11n_{153}$, $11n_{154}$, $11n_{158}$, $11n_{159}$, $11n_{160}$, $11n_{167}$, $11n_{168}$, $11n_{173}$, $11n_{176}$, $11n_{182}$, $11n_{183}$, $12a_{0001}$, $12a_{0008}$, $12a_{0011}$, $12a_{0013}$, $12a_{0015}$, $12a_{0016}$, $12a_{0020}$, $12a_{0024}$, $12a_{0026}$, $12a_{0030}$, $12a_{0033}$, $12a_{0048}$, $12a_{0058}$, $12a_{0060}$, $12a_{0066}$, $12a_{0070}$, $12a_{0077}$, $12a_{0079}$, $12a_{0080}$, $12a_{0091}$, $12a_{0099}$, $12a_{0101}$, $12a_{0111}$, $12a_{0115}$, $12a_{0119}$, $12a_{0134}$, $12a_{0139}$, $12a_{0141}$, $12a_{0142}$, $12a_{0146}$, $12a_{0157}$, $12a_{0184}$, $12a_{0186}$, $12a_{0188}$, $12a_{0190}$, $12a_{0209}$, $12a_{0214}$, $12a_{0217}$, $12a_{0219}$, $12a_{0222}$, $12a_{0245}$, $12a_{0246}$, $12a_{0250}$, $12a_{0261}$, $12a_{0265}$, $12a_{0268}$, $12a_{0271}$, $12a_{0281}$, $12a_{0299}$, $12a_{0316}$, $12a_{0323}$, $12a_{0331}$, $12a_{0333}$, $12a_{0334}$, $12a_{0349}$,

More non bi-orderable knot groups

$12a_{0351}$, $12a_{0362}$, $12a_{0363}$, $12a_{0369}$, $12a_{0374}$, $12a_{0386}$, $12a_{0396}$, $12a_{0398}$, $12a_{0426}$, $12a_{0439}$, $12a_{0452}$, $12a_{0464}$, $12a_{0466}$, $12a_{0469}$, $12a_{0473}$, $12a_{0476}$, $12a_{0477}$, $12a_{0479}$, $12a_{0497}$, $12a_{0499}$, $12a_{0515}$, $12a_{0536}$, $12a_{0561}$, $12a_{0565}$, $12a_{0569}$, $12a_{0576}$, $12a_{0579}$, $12a_{0629}$, $12a_{0662}$, $12a_{0696}$, $12a_{0697}$, $12a_{0699}$, $12a_{0700}$, $12a_{0706}$, $12a_{0707}$, $12a_{0716}$, $12a_{0815}$, $12a_{0824}$, $12a_{0835}$, $12a_{0859}$, $12a_{0864}$, $12a_{0867}$, $12a_{0878}$, $12a_{0898}$, $12a_{0916}$, $12a_{0928}$, $12a_{0935}$, $12a_{0981}$, $12a_{0984}$, $12a_{0999}$, $12a_{1002}$, $12a_{1013}$, $12a_{1027}$, $12a_{1047}$, $12a_{1065}$, $12a_{1076}$, $12a_{1105}$, $12a_{1114}$, $12a_{1120}$, $12a_{1122}$, $12a_{1128}$, $12a_{1168}$, $12a_{1176}$, $12a_{1188}$, $12a_{1203}$, $12a_{1219}$, $12a_{1220}$, $12a_{1221}$, $12a_{1226}$, $12a_{1227}$, $12a_{1230}$, $12a_{1238}$, $12a_{1246}$, $12a_{1248}$, $12a_{1253}$, $12n_{0005}$, $12n_{0006}$, $12n_{0007}$, $12n_{0010}$, $12n_{0016}$, $12n_{0019}$, $12n_{0020}$, $12n_{0038}$, $12n_{0041}$, $12n_{0042}$, $12n_{0052}$, $12n_{0064}$, $12n_{0070}$, $12n_{0073}$, $12n_{0090}$, $12n_{0091}$, $12n_{0092}$, $12n_{0098}$, $12n_{0104}$, $12n_{0105}$, $12n_{0106}$, $12n_{0113}$, $12n_{0115}$, $12n_{0120}$, $12n_{0121}$, $12n_{0125}$, $12n_{0135}$,

More non bi-orderable knot groups

$12n_{0136}$, $12n_{0137}$, $12n_{0139}$, $12n_{0142}$, $12n_{0148}$, $12n_{0150}$, $12n_{0151}$, $12n_{0156}$, $12n_{0157}$, $12n_{0165}$, $12n_{0174}$, $12n_{0175}$, $12n_{0184}$, $12n_{0186}$, $12n_{0187}$, $12n_{0188}$, $12n_{0190}$, $12n_{0192}$, $12n_{0198}$, $12n_{0199}$, $12n_{0205}$, $12n_{0226}$, $12n_{0230}$, $12n_{0233}$, $12n_{0235}$, $12n_{0242}$, $12n_{0261}$, $12n_{0272}$, $12n_{0276}$, $12n_{0282}$, $12n_{0285}$, $12n_{0296}$, $12n_{0309}$, $12n_{0318}$, $12n_{0326}$, $12n_{0327}$, $12n_{0328}$, $12n_{0329}$, $12n_{0344}$, $12n_{0346}$, $12n_{0347}$, $12n_{0348}$, $12n_{0350}$, $12n_{0352}$, $12n_{0354}$, $12n_{0355}$, $12n_{0362}$, $12n_{0366}$, $12n_{0371}$, $12n_{0372}$, $12n_{0377}$, $12n_{0390}$, $12n_{0392}$, $12n_{0401}$, $12n_{0402}$, $12n_{0405}$, $12n_{0409}$, $12n_{0416}$, $12n_{0417}$, $12n_{0423}$, $12n_{0425}$, $12n_{0426}$, $12n_{0427}$, $12n_{0437}$, $12n_{0439}$, $12n_{0449}$, $12n_{0451}$, $12n_{0454}$, $12n_{0456}$, $12n_{0458}$, $12n_{0459}$, $12n_{0460}$, $12n_{0466}$, $12n_{0468}$, $12n_{0472}$, $12n_{0475}$, $12n_{0484}$, $12n_{0488}$, $12n_{0495}$, $12n_{0505}$, $12n_{0506}$, $12n_{0508}$, $12n_{0514}$, $12n_{0517}$, $12n_{0518}$, $12n_{0522}$, $12n_{0526}$, $12n_{0528}$, $12n_{0531}$, $12n_{0538}$,

More non bi-orderable knot groups

$12n_{0543}$, $12n_{0549}$, $12n_{0555}$, $12n_{0558}$, $12n_{0570}$, $12n_{0574}$, $12n_{0577}$, $12n_{0579}$, $12n_{0582}$, $12n_{0591}$, $12n_{0592}$, $12n_{0598}$, $12n_{0601}$, $12n_{0604}$, $12n_{0609}$, $12n_{0610}$, $12n_{0613}$, $12n_{0619}$, $12n_{0621}$, $12n_{0623}$, $12n_{0627}$, $12n_{0629}$, $12n_{0634}$, $12n_{0640}$, $12n_{0641}$, $12n_{0642}$, $12n_{0647}$, $12n_{0649}$, $12n_{0657}$, $12n_{0658}$, $12n_{0660}$, $12n_{0666}$, $12n_{0668}$, $12n_{0670}$, $12n_{0672}$, $12n_{0673}$, $12n_{0675}$, $12n_{0679}$, $12n_{0681}$, $12n_{0683}$, $12n_{0684}$, $12n_{0686}$, $12n_{0688}$, $12n_{0690}$, $12n_{0694}$, $12n_{0695}$, $12n_{0697}$, $12n_{0703}$, $12n_{0707}$, $12n_{0708}$, $12n_{0709}$, $12n_{0711}$, $12n_{0717}$, $12n_{0719}$, $12n_{0721}$, $12n_{0725}$, $12n_{0730}$, $12n_{0739}$, $12n_{0747}$, $12n_{0749}$, $12n_{0751}$, $12n_{0754}$, $12n_{0761}$, $12n_{0762}$, $12n_{0781}$, $12n_{0790}$, $12n_{0791}$, $12n_{0798}$, $12n_{0802}$, $12n_{0803}$, $12n_{0835}$, $12n_{0837}$, $12n_{0839}$, $12n_{0842}$, $12n_{0848}$, $12n_{0850}$, $12n_{0852}$, $12n_{0866}$, $12n_{0871}$, $12n_{0887}$, $12n_{0888}$.

Theorem: All roots positive implies bi-orderable

As motivation, consider an upper triangular matrix multiplied by a vector:

$$\begin{pmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 + *x_2 + *x_3 \\ \lambda_2 x_2 + *x_3 \\ \lambda_3 x_3 \end{pmatrix}$$

Now, declaring a vector (in $\mathbb{R}^3$) to be "positive" if its last nonzero entry is greater than zero, we see that, if also the eigenvectors λ_i are positive, then multiplication by such a matrix preserves that positive cone of $\mathbb{R}^3$, considered as an additive group. So we see

Proposition

If all the eigenvalues of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ are real and positive, then there is a bi-ordering of $\mathbb{R}^n$ which is preserved by L.

Theorem: All roots positive implies bi-orderable

So our problem reduces to showing:

Proposition

Let F be a finitely generated free group and $h : F \to F$ an automorphism. If all the eigenvalues of $h_* : H_1(F; \mathbb{Q}) \to H_1(F; \mathbb{Q})$ are real and positive, then there is a bi-ordering of F preserved by h.

One way to order a free group F is to use the lower central series $F_1 \supset F_2 \supset \cdots$ defined by

$$F_1 = F, \quad F_{i+1} = [F, F_i]$$

which has the properties that $\bigcap F_i = \{1\}$ and F_i/F_{i+1} is free abelian. Choose an arbitrary bi-ordering of F_i/F_{i+1}, and define a positive cone of F by declaring $1 \neq x \in F$ positive if its class in F_i/F_{i+1} is positive in the chosen ordering, where i is the last subscript such that $x \in F_i$.

Theorem: All roots positive implies bi-orderable

If $h : F \to F$ is an automorphism it preserves the lower central series and induces maps of the lower central quotients: $h_i : F_i/F_{i+1} \to F_i/F_{i+1}$. With this notation, h_1 is just the abelianization h_{ab}. In a sense, all the h_i are determined by h_1. That is, there is an embedding of F_i/F_{i+1} in the tensor power $F_{ab}^{\otimes k}$, and the map h_i is just the restriction of $h_{ab}^{\otimes k}$.

The assumption that all eigenvalues of h_{ab} are real and positive implies that the same is true of all its tensor powers.

This allows us to find bi-orderings of the free abelian groups F_i/F_{i+1} which are invariant under h_i. Using these to bi-order F, we get invariance under h, which proves the proposition and therefore the theorem.

Theorem: Bi-orderable implies some positive roots

We now turn to the proof of the theorem: If K is fibred and its group is bi-orderable, then $\Delta_K(t)$ has some real positive roots.

Theorem

Suppose G is a nontrivial finitely generated bi-orderable group and that $\phi : G \to G$ preserves a bi-ordering of G. Then the induced map

$$\phi_* : H_1(G;\mathbb{Q}) \to H_1(G;\mathbb{Q})$$

has a positive eigenvalue.

Theorem: Bi-orderable implies some positive roots

The key idea is to consider a linear automorphism $L : \mathbb{Q}^n \to \mathbb{Q}^n$ which preserves an ordering. Regarding $\mathbb{Q}^n$ as a subset of $\mathbb{R}^n$, there is a hyperplane $H \subset \mathbb{R}^n$ defined by

$H = \{x \in \mathbb{R}^n |$ every nbhd. of x contains positive and negative points$\}$

H separates $\mathbb{R}^n$ and is invariant under L.

Consider the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$, and let D denote the closed hemisphere of $\mathbb{S}^{n-1}$ which lies on the "positive" side of H. There is a mapping $D \to D$ defined by

$$x \to \frac{L(x)}{|L(x)|}$$

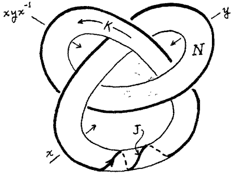
Since D is an $(n-1)$-ball, this map has a fixed point (Brouwer). This fixed point corresponds to an eigenvector of L, which has a positive real eigenvalue.

Surgery

We conclude with some applications to surgery on a knot K in $\mathbb{S}^3$. One removes a tubular neighborhood of K and attaches a solid torus $\mathbb{S}^1 \times D^2$ so that the meridian $\{*\} \times \mathbb{S}^1$ is attached to a specified "framing" curve on the boundary of the neighborhood.

By theorem of Wallace and Lickorish, every compact, orientable 3-manifold (without boundary) can be constructed by surgery on some disjoint union of knots (i. e. a link) in $\mathbb{S}^3$.

Surgery



Consider surgery on the trefoil knot:

With $+1$ framing, as pictured, one gets the Poincaré homology sphere, as constructed by Max Dehn.This is a homology sphere with fundamental group

$$\langle a, b | (ab)^2 = a^3 = b^5 \rangle$$

This is a finite group, of order 120,
so its group is certainly not left-orderable.

Surgery

For the next example, we'll need to consider $\widetilde{SL_2}(\mathbb{R})$, which is one of the eight Thurston 3-manifold geometries.
Note that the matrix group $SL_2(\mathbb{R})$ acts on the circle. For example, it acts on $S^1 \cong \mathbb{R} + \infty$ by fractional linear transformations, preserving orientation.
In fact as a topological space, $SL_2(\mathbb{R})$ has the homotopy type of the circle. Therefore its universal covering $\widetilde{SL_2}(\mathbb{R}) \to SL_2(\mathbb{R})$ is an infinite cyclic covering. Moreover, $\widetilde{SL_2}(\mathbb{R})$ has a group structure and acts on the universal cover $\mathbb{R}$ of $\mathbb{S}^1$ by orientation-preserving homeomorphisms.
That is, $\widetilde{SL_2}(\mathbb{R})$ is a subgroup of $Homeo_+(\mathbb{R})$, and we conclude that $\widetilde{SL_2}(\mathbb{R})$ is a left-orderable group.

Surgery

If we do surgery on the trefoil using -1 framing, the resulting 3-manifold M, again a homology sphere, has fundamental group

$$\langle a, b | (ab)^2 = a^3 = b^7 \rangle$$

G. Bergman observed that this group maps injectively to $\widetilde{SL_2}(\mathbb{R})$, which is a left-orderable group. Thus $\pi_1(M)$ is left-orderable (even though its first Betti number is zero).
It is not bi-orderable or even locally indicable, because it is finitely-generated and perfect (that is, abelianizes to the trivial group).

Surgery

Theorem

Suppose K is a fibred knot in S^3 and nontrivial surgery on K produces a 3-manifold M whose fundamental group is bi-orderable. Then the surgery must be longitudinal (that is, 0-framed) and $\Delta_K(t)$ must have a positive real root. Moreover, M fibres over S^1.

Surgery

Ozsváth and Szabó define an L-space to be a closed 3-manifold M such that $H_1(M;\mathbb{Q}) = 0$ and its Heegaard-Floer homology $\widehat{HF}(M)$ is a free abelian group of rank equal to $|H_1(M;\mathbb{Z})|$. Lens spaces, and more generally 3-manifolds with finite fundamental group are examples of L-spaces. But there are also many rational homology spheres whose fundamental group is infinite.

Theorem

Suppose $K \subset S^3$ is a knot whose group is bi-orderable. Then one cannot obtain an L-space by surgery on K.

Surgery

Proof sketch: Suppose surgery on K results in an L-space.
By Yi Ni, K must be fibred. Moreover, Ozsváth and Szabó show that the Alexander polynomial of K must have a special form.
Then one argues that a polynomial of this form has no positive real roots, so the knot group cannot be bi-ordered.

Merci beaucoup!

Next time: Braids, $Aut(F_n)$ and minimal volume hyperbolic 3-manifolds.