BRAIDS IN CONTACT STRUCTURES

BY

VERA VÉRTESI

PLAN OF TALKS

I. Introduction to contact structures

II. Bennequin's Thm
transverse braids
braid foliations

III. Open book decompositions

IV. Legendrian braids
convex surface theory

I. CONTACT STRUCTURES

Def A contact structure ξ^{2n} is a totally non-integrable hyperplane field on M^{2n+1}.

not tangent to any open hypersurface

$\Updownarrow$

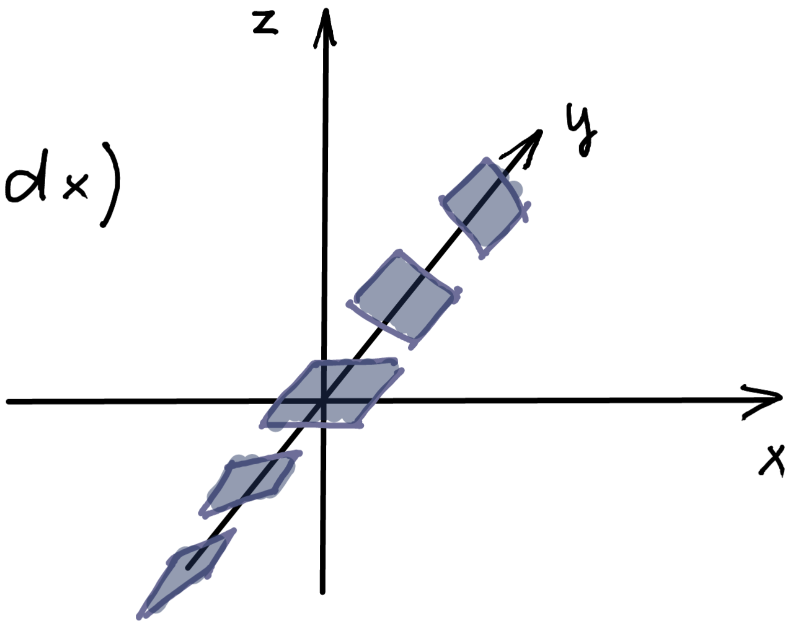
If $\xi = \ker \alpha$ then $\underbrace{\alpha \wedge d\alpha^{\wedge n}}_{(2n+1)\text{-form}} \neq 0$

e.g. $M = \mathbb{R}^{2n+1}$

$\xi_{st} = \ker\left(dz - \sum_{i=1}^{n} y_i \, dx_i\right)$

for $n=1$:

$\mathbb{R}^3$, $\xi_{st} = \ker(dz - y\,dx)$

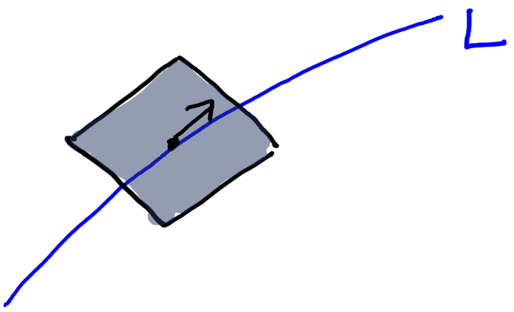


Thm (Darboux) Locally every contact structure looks like this

Def: $L^n \subseteq (M^{2n+1}, \xi)$ is a Legendrian submanifold if $T_pL \subseteq \xi_p \quad \forall x \in L$

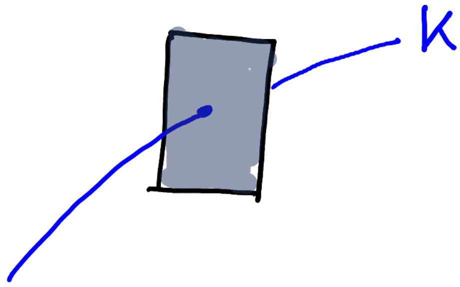
E.g: $\{x_1 = c_1, \dots, x_n = c_n, z = c\}$ planes in $(\mathbb{R}^{2n+1}, \xi_{st})$

$\boxed{n=3}$ (except in some of the examples)



Legendrian arc

$T_pL \in \xi_p$



transverse arc

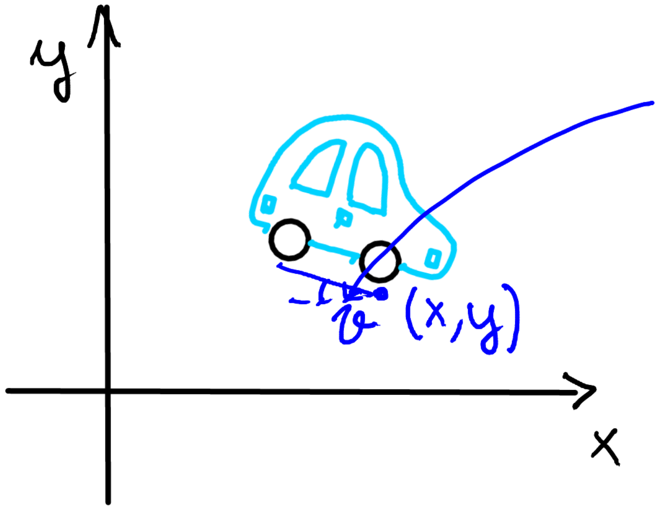
$T_pL \pitchfork \xi_p$

Thm (Legendrian approximation)

Any Legendrian arc can be C^0-approximated by a Legendrian arc

Legendrian knots in real life

Configuration space of the front wheel of a car



position , angle :

$\mathbb{R}^2 \times S^1$

$(x,y) \qquad \vartheta$

car goes when front wheel points :

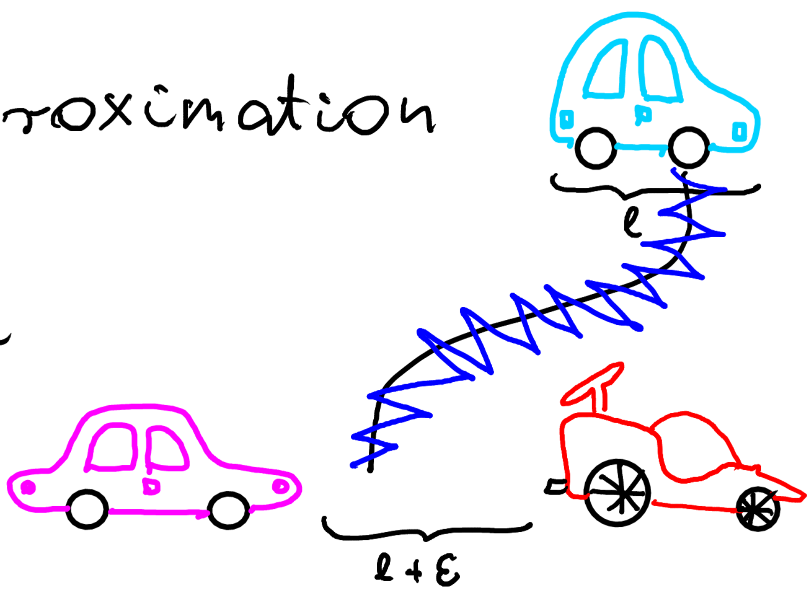
$$\frac{dy}{dx} = \operatorname{tg} \vartheta$$

$\rightsquigarrow$ take $\xi = \ker(\cos\vartheta \, dy + \sin\vartheta \, dx)$

motion of car $\longleftrightarrow$ Legendrian curves

Legendrian approximation

$\Rightarrow$ can always parallel park your car



Classical application of contact structures

- PDE (Sophus Lie 1872)

  $F : \mathbb{R}^{2n+1} \to \mathbb{R}$

  finding solutions $z : \mathbb{R}^n \to \mathbb{R}$

  of $$F(\underbrace{\underline{x}}_{\in \mathbb{R}^n}, \underbrace{z'(\underline{x})}_{\in \mathbb{R}^n}, \underbrace{z(\underline{x})}_{\in \mathbb{R}}) = 0$$

  is equivalent to finding $u : \mathbb{R}^n \to \mathbb{R}^{2n+1}$

  s.t.

  $$F(u) = 0$$

  & $\mathrm{Im}(u)$ is Legendrian in ξ_{st}

- Riemannian geometry g metric on M

  $$\begin{array}{ccc} TM & \underset{g}{\cong} & T^*M \\ \cup & & \cup \\ S(TM) & & S(T^*M) \;\; \xi \end{array}$$

  canonical ctct str.

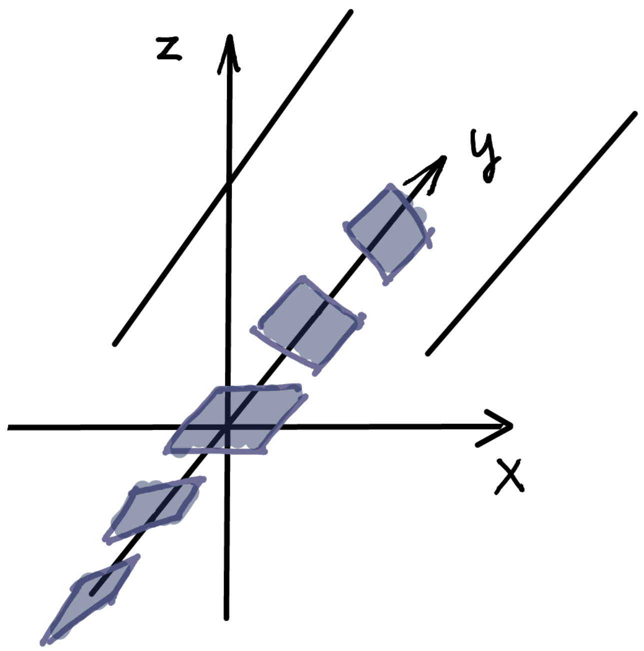
  geodesic flow $\leftrightarrow$ Reeb flow

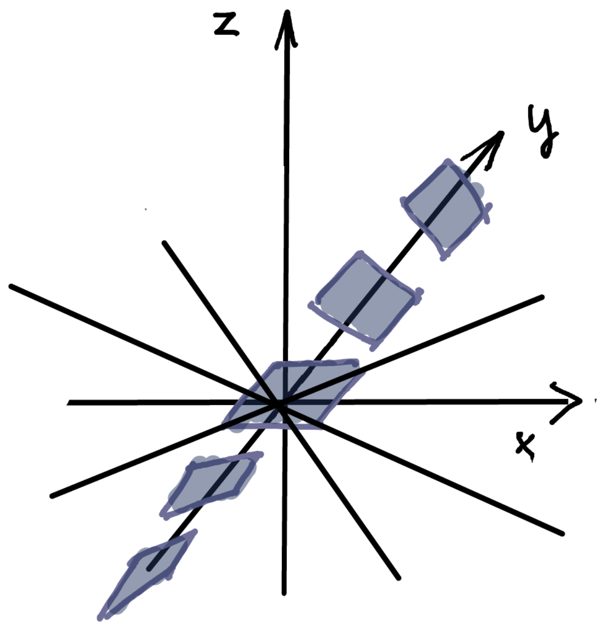
- optics (Huygens)
- Thermodynamics (Gibbs)

Lot of modern applications!

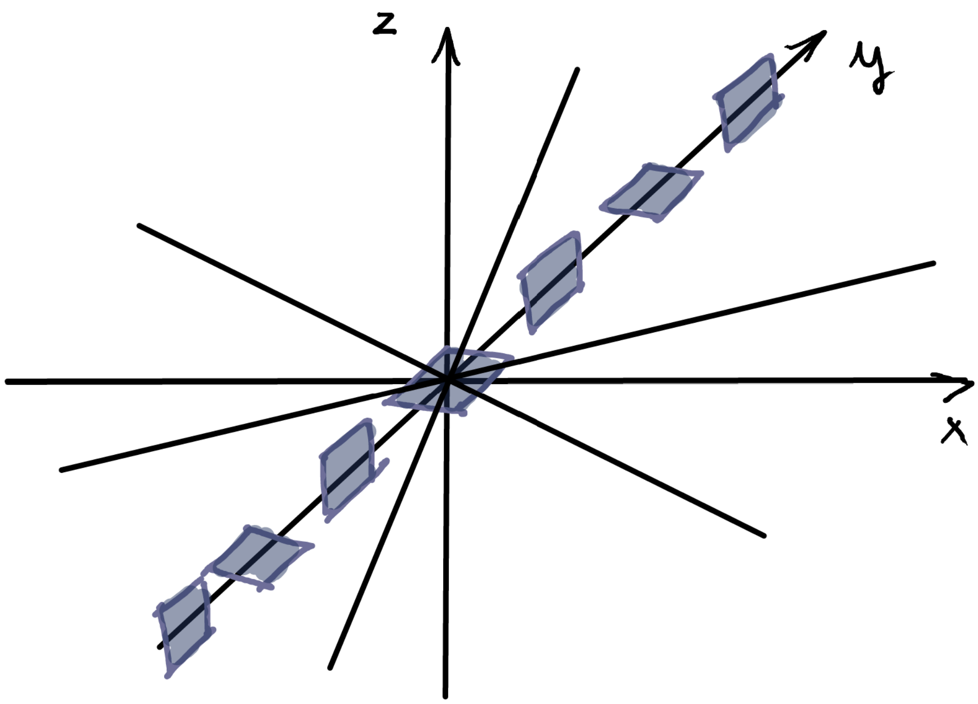
II. BENNEQUIN'S THEOREM

contact structures on $\mathbb{R}^3$





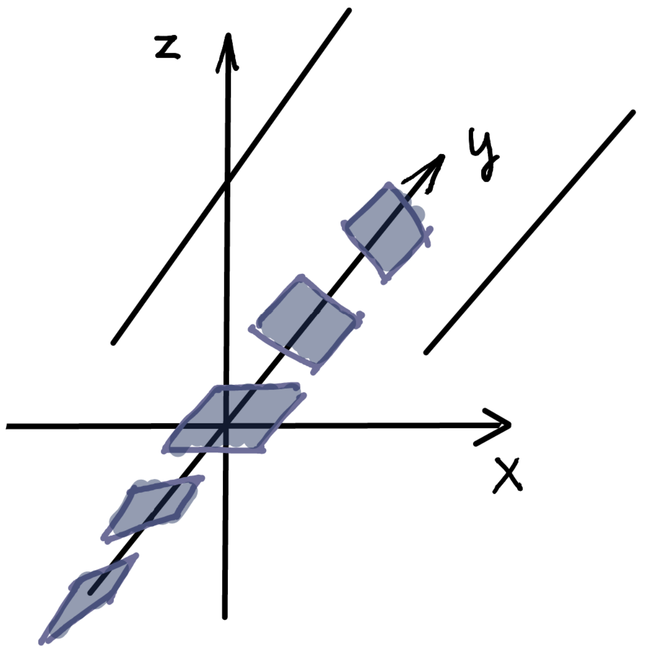
$\xi_{st} = \ker(dz - y\,dx)$ $\quad$ $\xi_{sym} = \ker(dz + r^2 d\vartheta)$

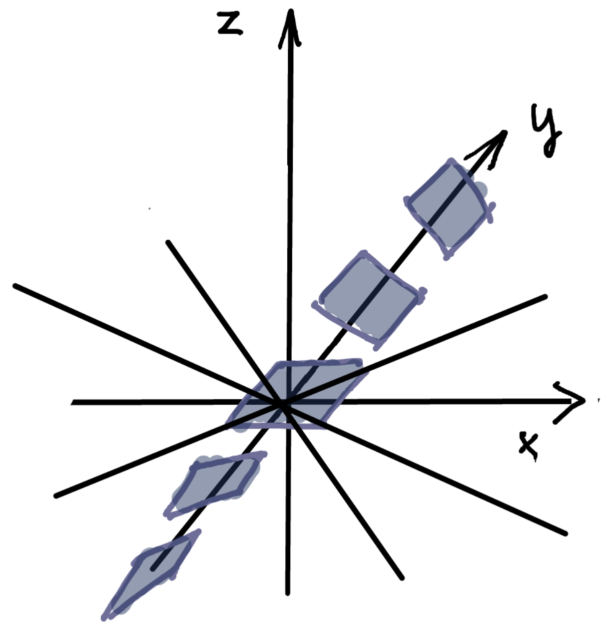


$\xi_{ot} = \ker(\cos r\,dz + r\sin r\,d\vartheta)$

HW Check that they are indeed contact structures!

Question: Are ξ_{st}, ξ_{sym}, ξ_{ot} different?





$\xi_{st} = \ker(dz - y\,dx)$ $\qquad$ $\xi_{sym} = \ker(dz + r^2 d\vartheta)$

$$dz + r^2 d\vartheta \;=\; dz + x\,dy - y\,dx$$

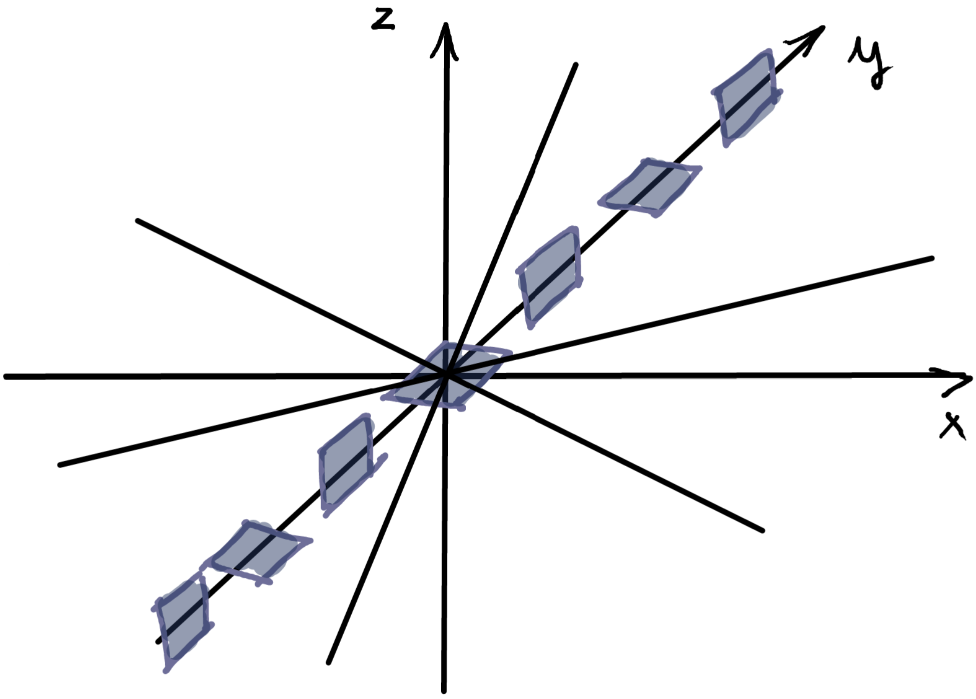
$\Rightarrow$ ξ_{st} & ξ_{sym} are **isotopic**:

$$\xi_{st} \xrightarrow{\quad \xi_t := \ker(dz - y\,dx + t\,x\,dy) \quad} \xi_{sym}$$

Question: Is ξ_{sym} & ξ_{ot} isotopic?

Is ξ_{sym} & ξ_{ot} **contactomorphic**?

? $\exists$ $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ diffeomorphism w/ $\varphi_* \xi_{sym} = \xi_{ot}$



$\xi_{ot} = \ker(\cos r\, dz + r \sin r\, d\theta)$

Bennequin: $\xi_{sym} \not\simeq \xi_{OT}$

Plan of proof: use transverse knots

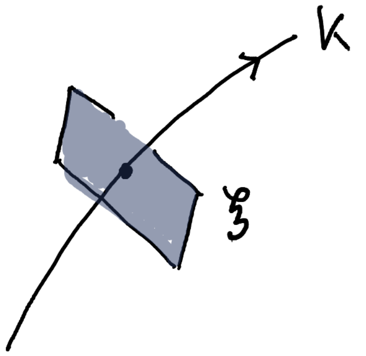
① Define an invariant for transverse knots:
self linking number : $sl(K)$

② Prove $sl(K) \leq -\chi(\Sigma)$ in ξ_{sym}

Seifert surface for K

③ Show $sl(U) = 1 > -1$ in ξ_{OT}

Remember:



transverse knot

$K \pitchfork \xi$

① Self linking number

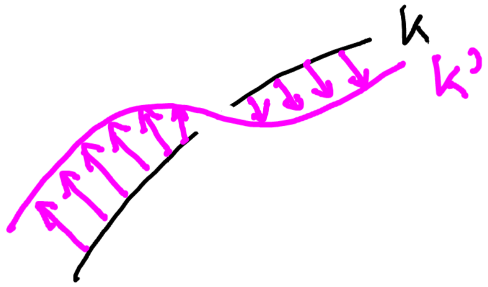
any contact structure

nonzero section

$$\xi \underset{v \neq 0}{\overset{\downarrow}{\rightleftarrows}} \mathbb{R}^3$$

[obstruction: $e(\xi) \in H^2(\mathbb{R}^3; \mathbb{Z}) = 0$]

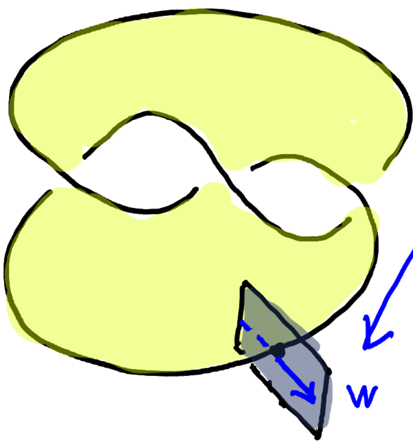
Def: K transverse knot, let $K' = K + \varepsilon v$



$$sl(K) = lk(K, K')$$

HW: $sl(K)$ is independent of the choice of v

HW: $sl(K) = \langle e(\xi, w), [\Sigma] \rangle$

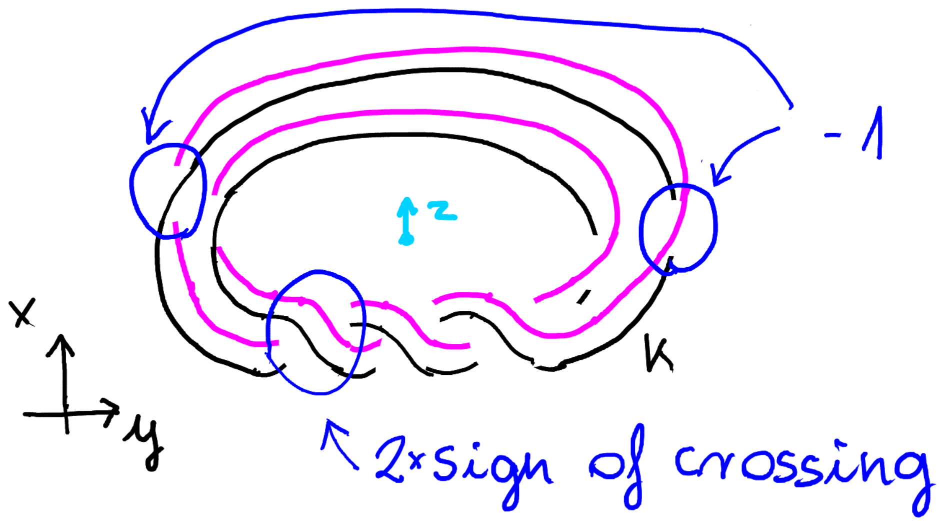


$T\Sigma \cap \xi$
pointing "out"
of Σ

Towards ② & ③: Computing $sl(K)$:

• in ξ_{sym}: (HW): $v = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ is a section

K is the closure of a braid B



braid index

$$\Rightarrow sl(K) = lk(K, K') - \frac{1}{2}\left(\begin{matrix}\text{signed} \\ \text{intersections}\end{matrix}\right) = a(B) - n(B)$$

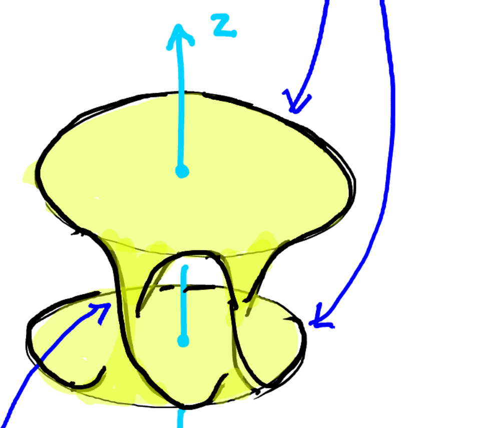
algebraic length / exponent sum:

$B = \prod \sigma_{n_i}^{\varepsilon_i}$ $\qquad a(B) = \sum \varepsilon_i$

Rmk: B positive braid

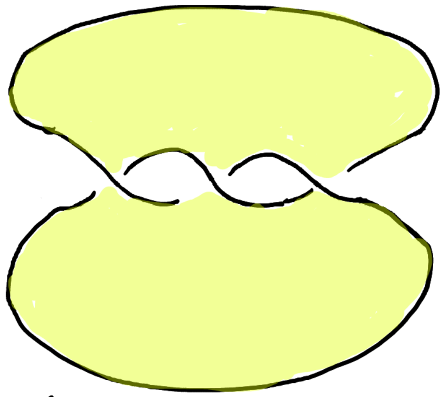
$\Rightarrow \exists$ Seifert surface w/

$sl(K) = -\chi(\Sigma)$

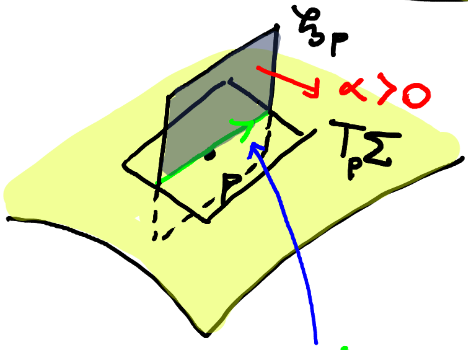


▸ Computing sl(K) using a Seifert surface

- Pick Σ w/ $\partial \Sigma = K$



- We can define an (oriented) linefield $l_p = \xi_p \cap T_p \Sigma$:



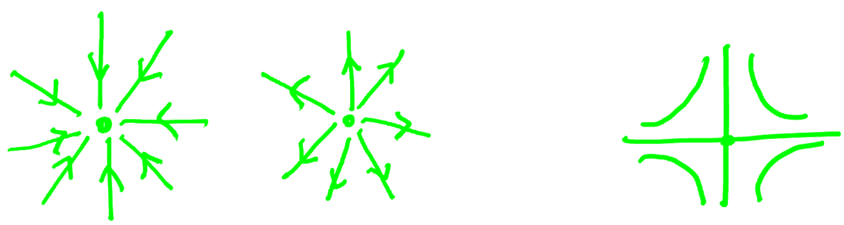
$(\alpha > 0, \vec{l}_p)$ positive basis for $T_p \Sigma$

- by isotoping Σ can make sure only finitelly many singularities:
  - $+$ if orientation of $T_p \Sigma$ & ξ_p agree
  - $-$ if disagree

Def: the integral curves of $\vec{l}_p$ is the characteristic foliation of Σ: $\mathcal{F}_\xi$

singularities:

$+$ $-$



elliptic hyperbolic

Let $e_\pm = \#(\pm$ elliptic points of $\mathcal{F}_\xi)$

$h_\pm = \#(\pm$ hyperbolic points of $\mathcal{F}_\xi)$

Poincaré - Hopf: $\chi(\Sigma) = (e_+ + e_-) - (h_+ + h_-)$

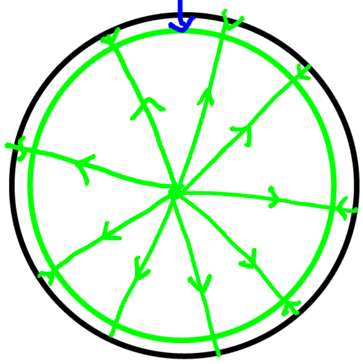
HW $\boxed{sl(K) = -(e_+ - h_+) + (e_- - h_-)}$

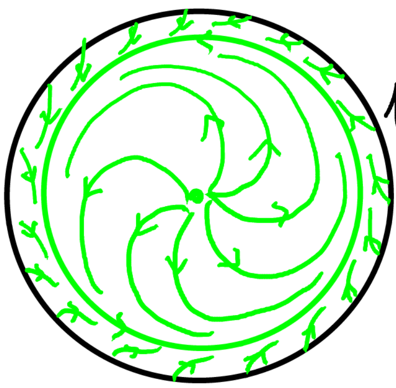
③ The "overtwisted disc":

$\xi_{OT} = \ker(\cos r \, dz + r \sin r \, d\vartheta)$

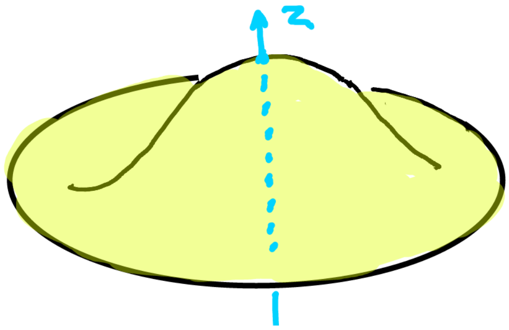
- Take $D^2 = \{z = 0, r \leq \pi + \varepsilon\}$
- Push up the middle slightly

singular circle





$\mathcal{U}$



$e_- = h_\pm = 0$

$e_+ = 1$

$\Rightarrow sl(\mathcal{U}) = 1 > -\chi(D^2) = -1$

Continueing towards ②

Remember: $sl(K) = -e_+ + e_- - h_+ - h_-$

Poincaré - Hopf: $\chi(K) = (e_+ + e_-) - (h_+ + h_-)$

$\rightsquigarrow \quad sl(K) \overset{?}{\leq} -\chi(K) \quad \Leftrightarrow \quad e_- \overset{?}{\leq} h_-$

$\Rightarrow$ **Strategy:** isotop Σ so that $e_- = 0$

We will use another foliation

braid foliation $\mathcal{F}_b$

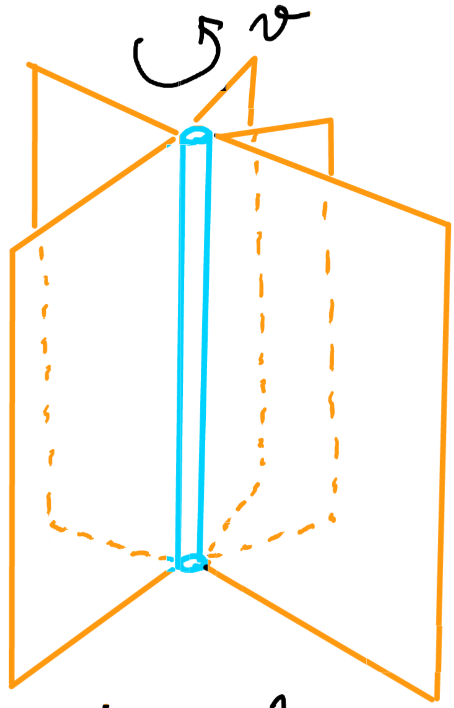
a) "close" to $\mathcal{F}_\xi$ (See: later)

$\Rightarrow$ can be used to compute $sl(K)$

b) more rigid than $\mathcal{F}_\xi \Rightarrow$ easier to understand & keep track of changes

Braid foliation

$$\begin{array}{c} \mathbb{R}^3 - \{z\text{-axis}\} \\ \big\downarrow \theta \\ S^1 \end{array}$$

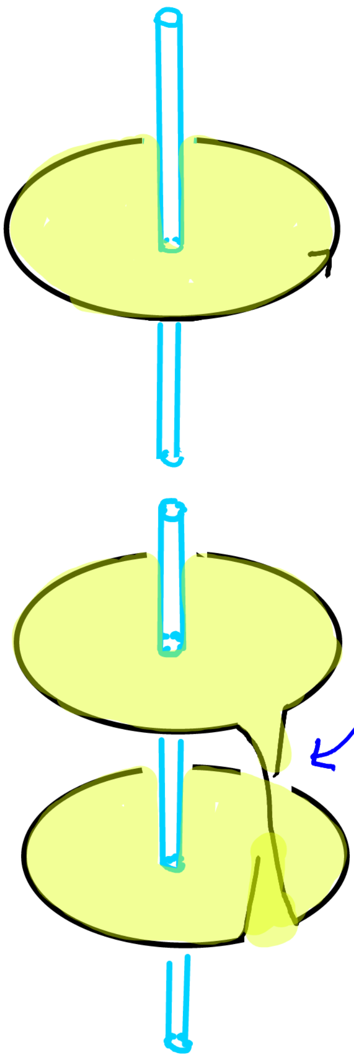


$H_{\theta_0} = \{\theta = \theta_0\}$

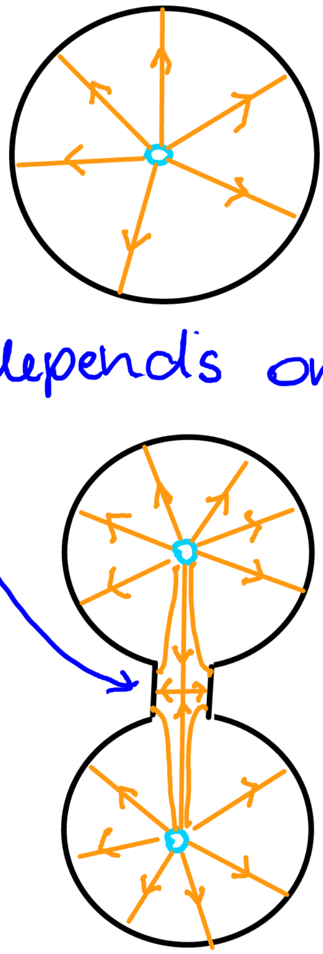
as for $\mathcal{F}_\xi$

Def: the oriented foliation induced by $\Sigma \cap H_\theta$ is the braid foliation $\mathcal{F}_b$ of Σ.

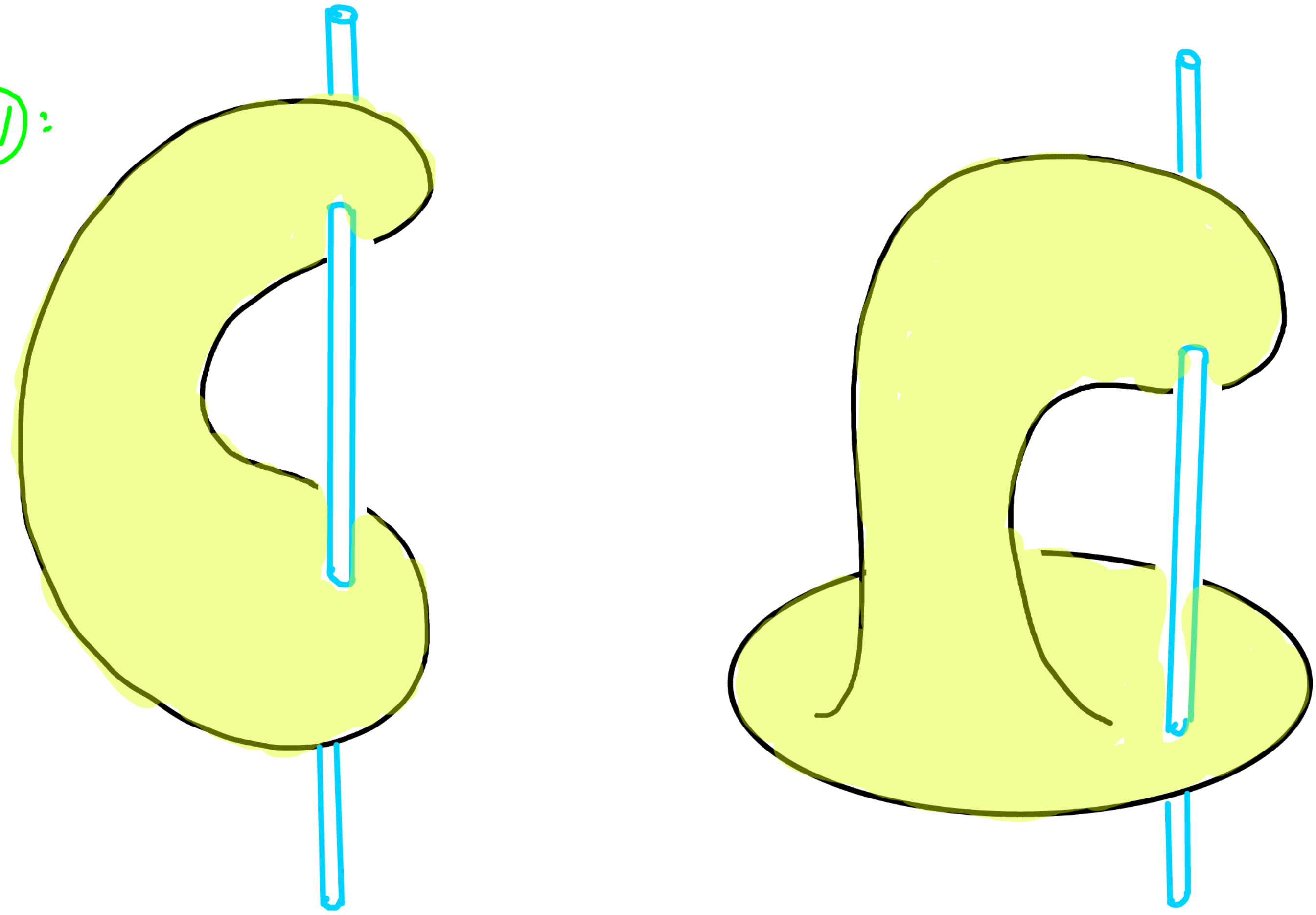
Example:



sign depends on twist

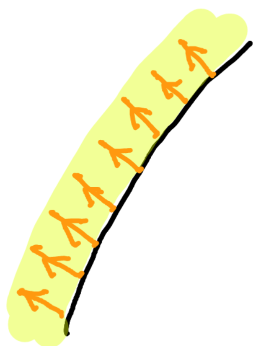


HW:



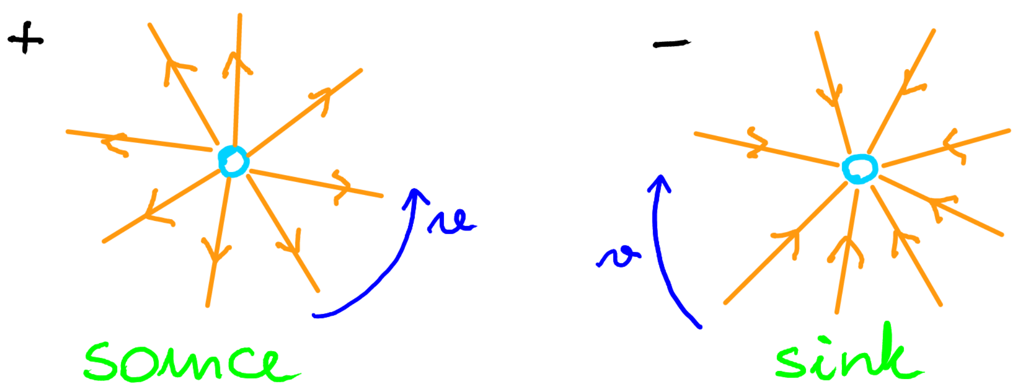
Properties of $\mathcal{F}_b$

- near $\partial\Sigma = K$



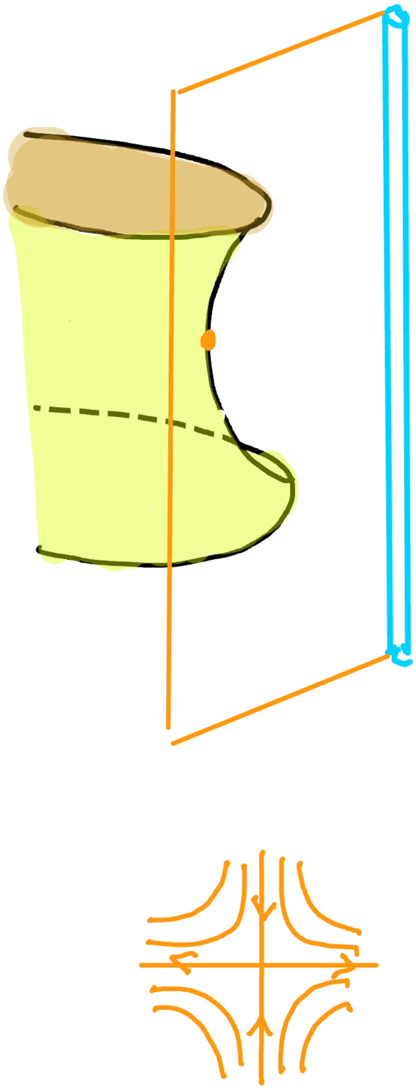
- Can assume $\Sigma \pitchfork \{z\text{-axis}\}$

  $\Rightarrow$ finitelly many elliptic points

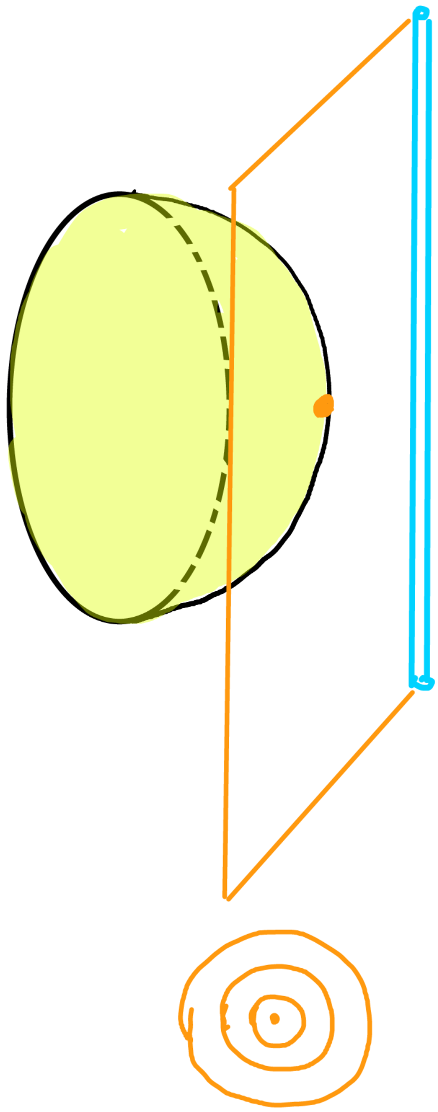


- By C^∞-small isotopy we can arrange that $\vartheta|_\Sigma$ is a circle-valued Morse function (away from the $\{z$-axis$\}$)

$\Rightarrow$ We have two types of singularities:



saddle/hyperbolic

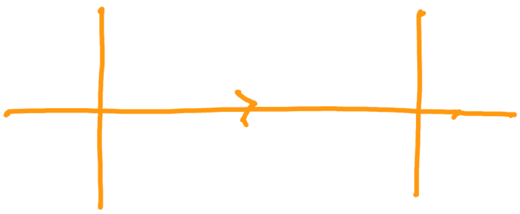


center

HW How to recognise +/- saddles?

- Moreover we can assume that on each H_v we have at most 1 critical point

$\Rightarrow$ no saddle-saddle connection

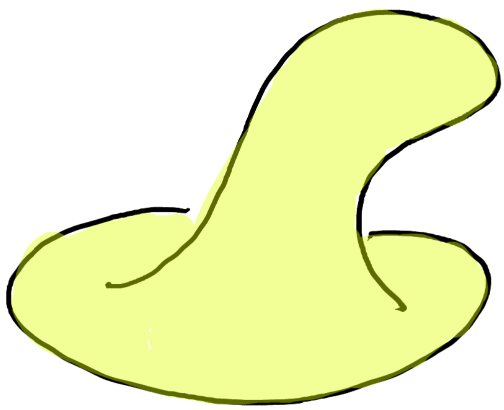


$\swarrow$ these all happen in the same v

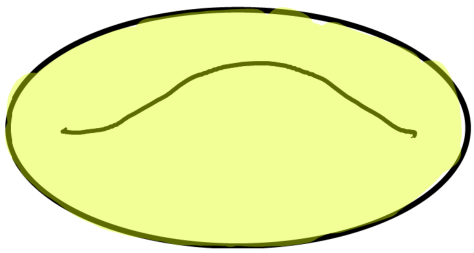
Removing centers:

expanding along a center:

in $\mathbb{R}^3$ $\longrightarrow$



$\wr$ isotopy

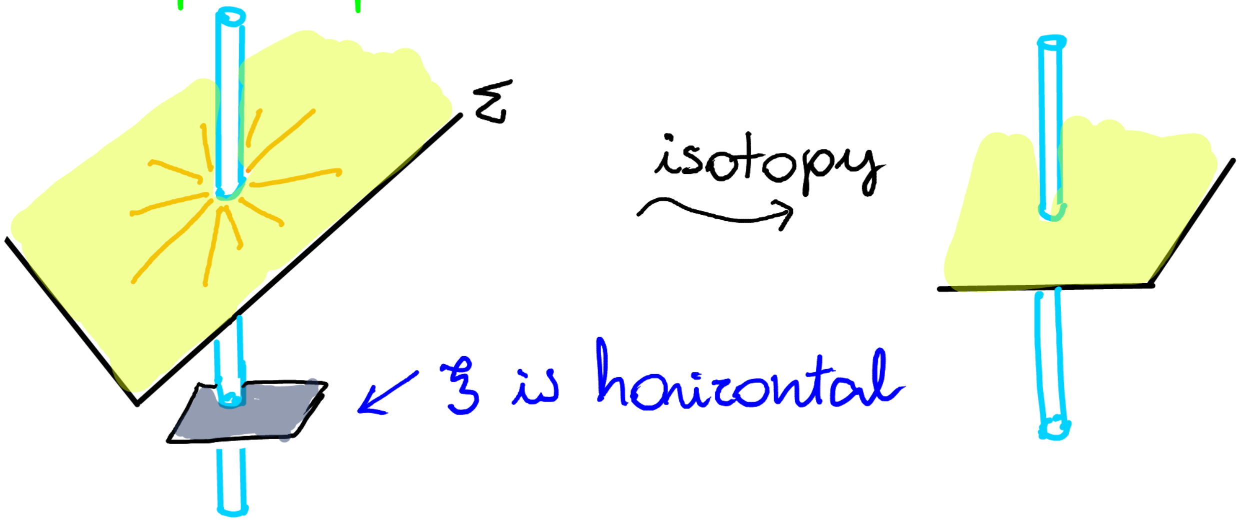


$\xleftarrow{\mathcal{F}_6}$

Comparing $\mathcal{F}_b$ to $\mathcal{F}_\xi$

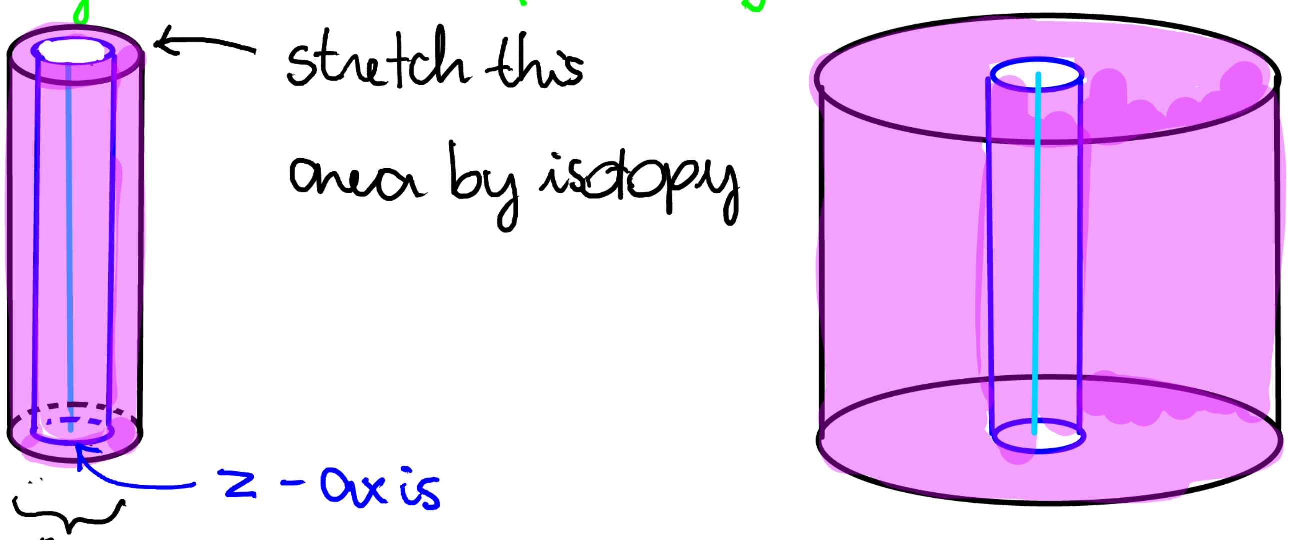
$\xi_{sym} = \ker\left(dz + r^2\, d\vartheta\right)$

→ elliptic points



$\Rightarrow$ $\mathcal{F}_b = \mathcal{F}_\xi$ along the $\{z\text{-axis}\}$

→ away from the $\{z\text{-axis}\}$



$\mathcal{F}_b = \mathcal{F}_\xi$ here, only elliptic points

$\Longrightarrow$ $r \gg R$ $\quad \xi_{sym} \approx H_\vartheta \rightarrow \mathcal{F}_b = \mathcal{F}_\xi$ everywhere

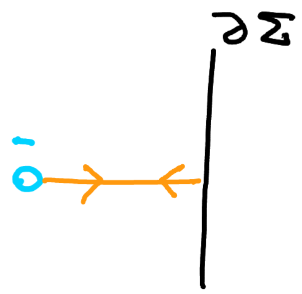
HW Where did we use that $\mathcal{F}_b$ has no centers?

Removing negative elliptic points

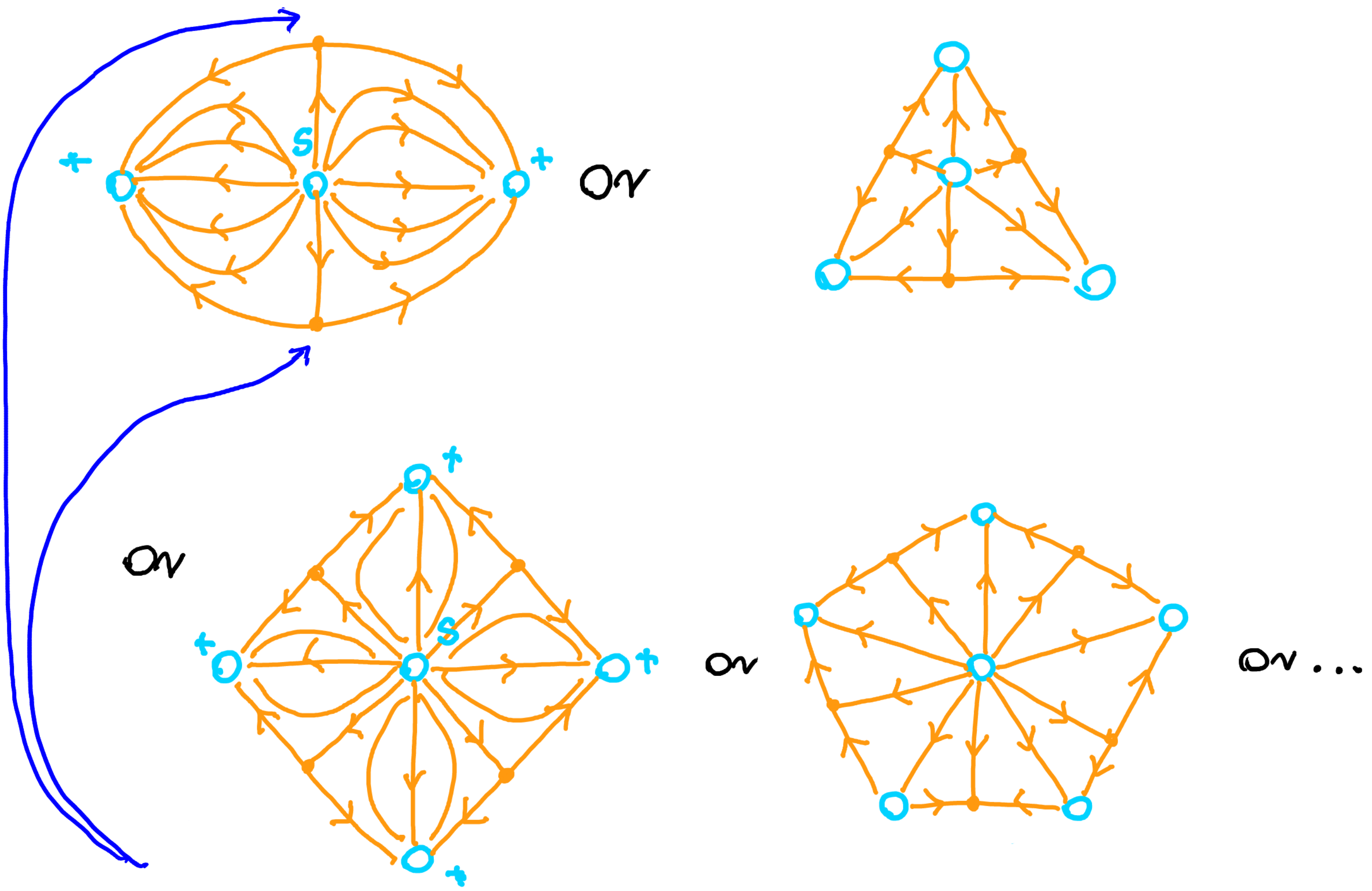
s – negative elliptic point

Def: the closure of union of leaves limiting to s is the **star** of s: D_s

- D_s is disjoint from $\partial\Sigma$:



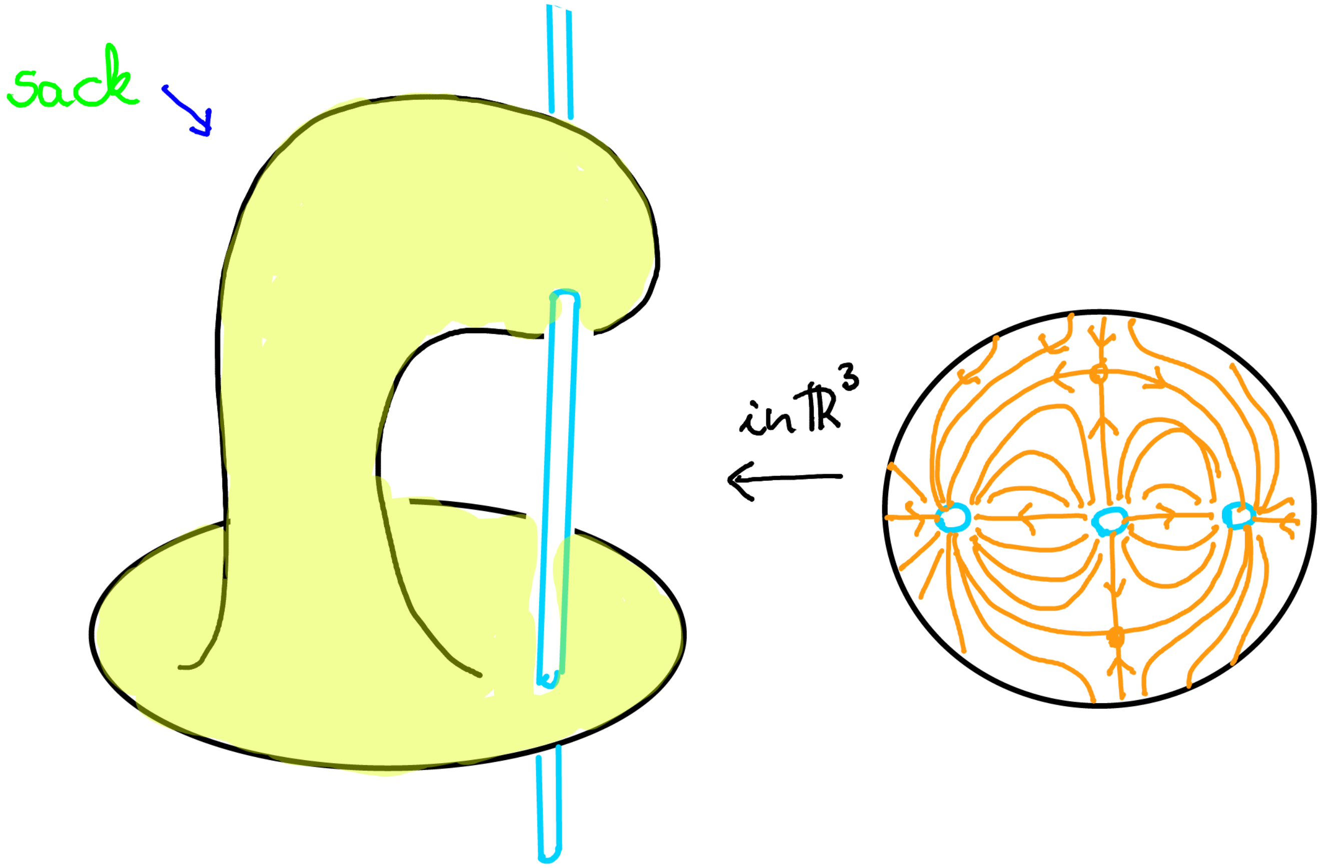
- D_s looks like:



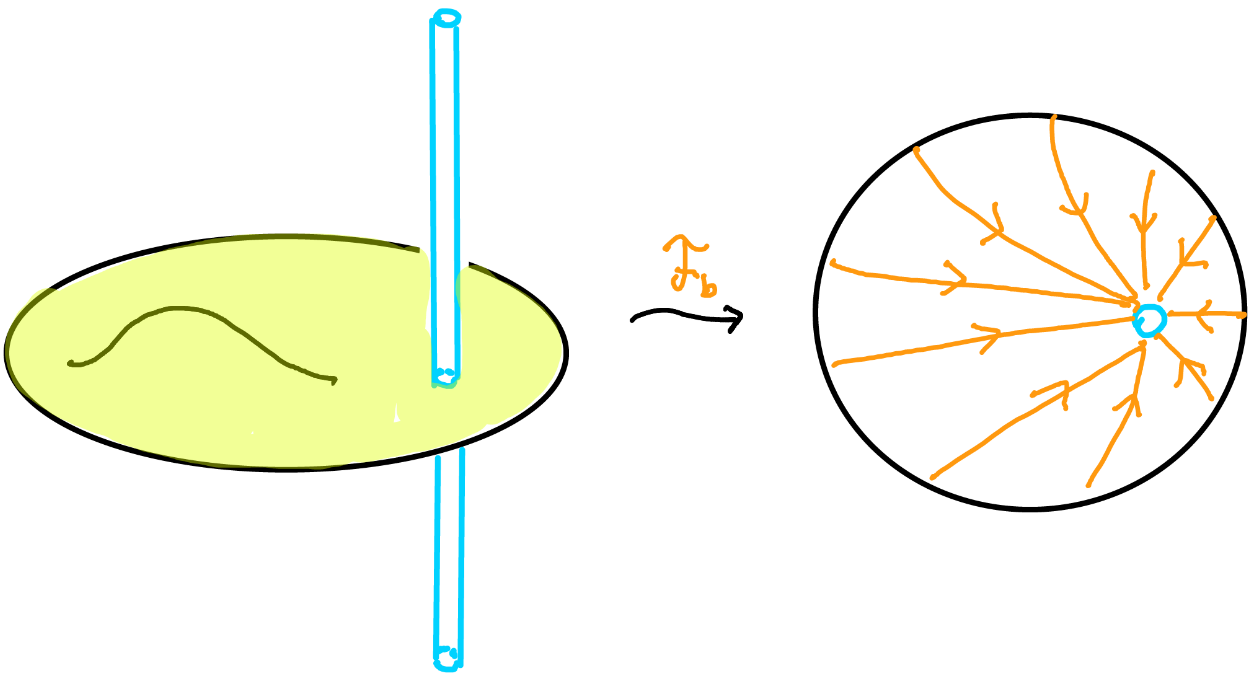
these all happen at different ν

$\Rightarrow D_s$ is embedded

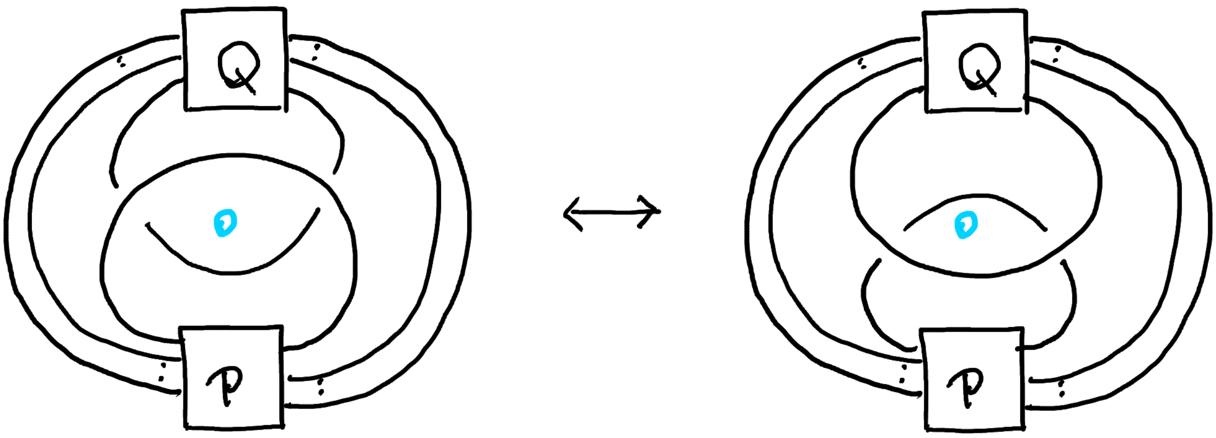
Removal of a bigon



if empty $\Rightarrow$ by isotopy

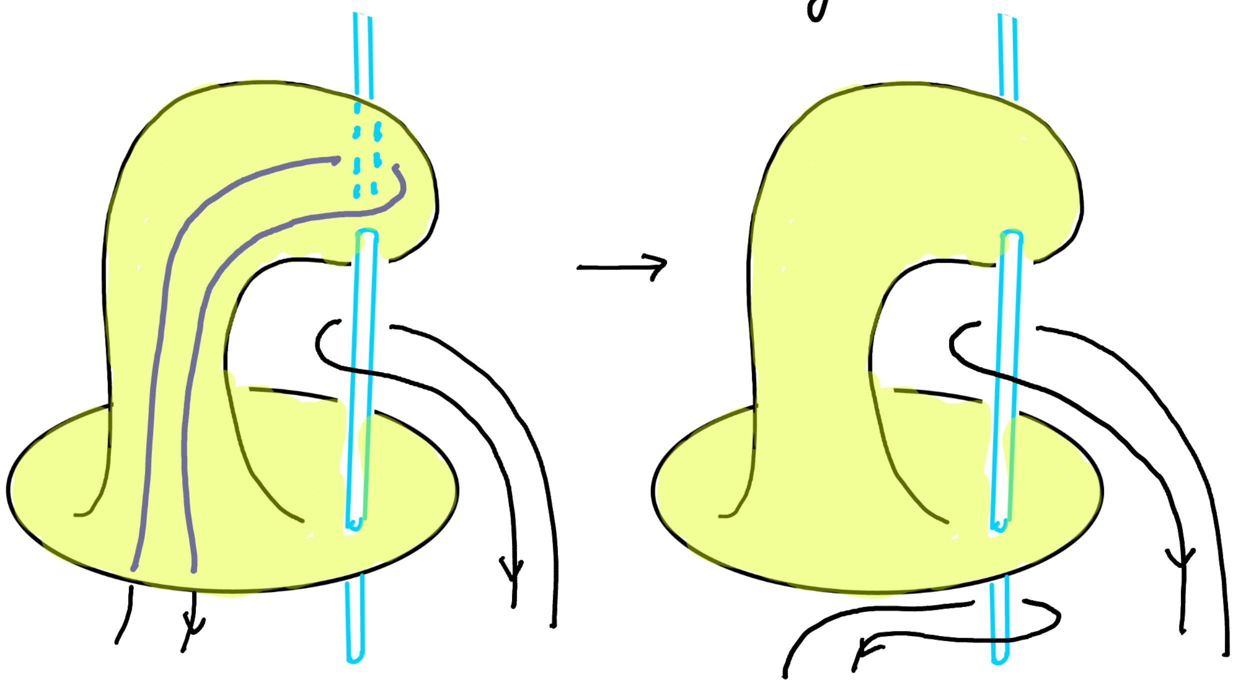


if the sack is not empty
$\Rightarrow$ can emty it out by:
exchange moves:



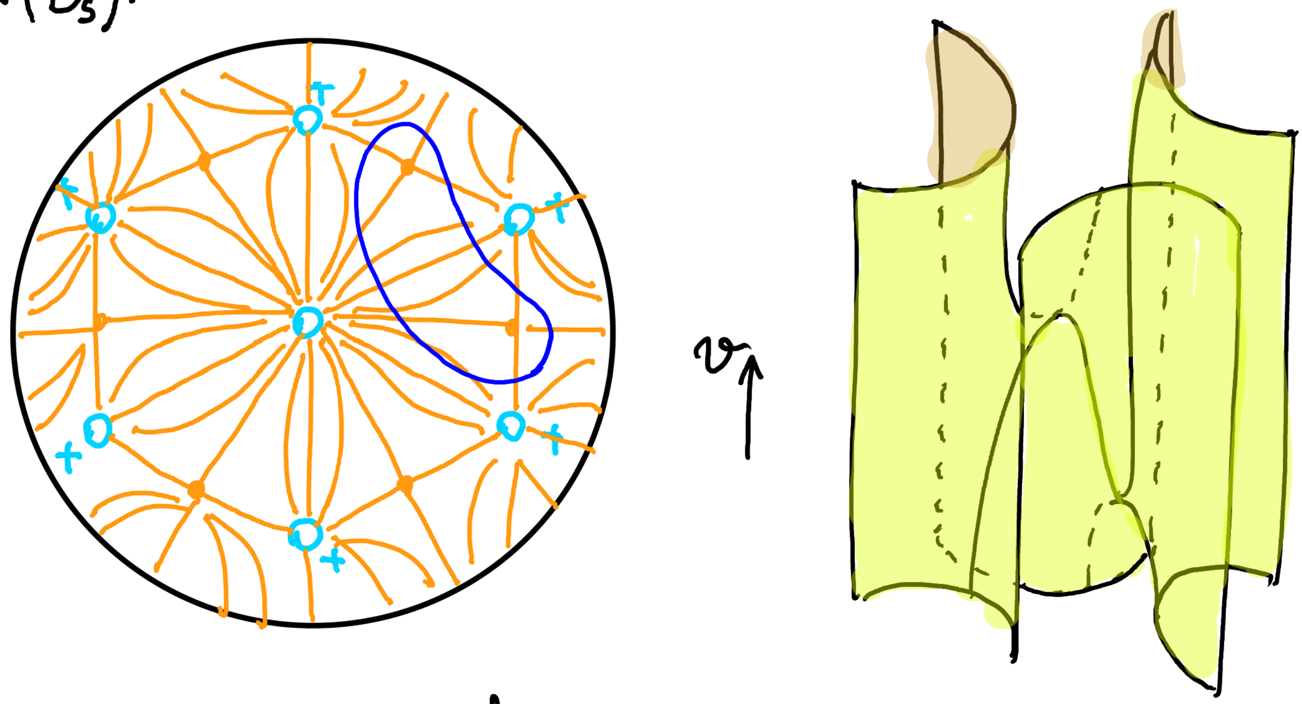
HW exchange move induces transverse isotopy

enough: • doesn't change a or n
$\Rightarrow$ doesn't change sl(K)

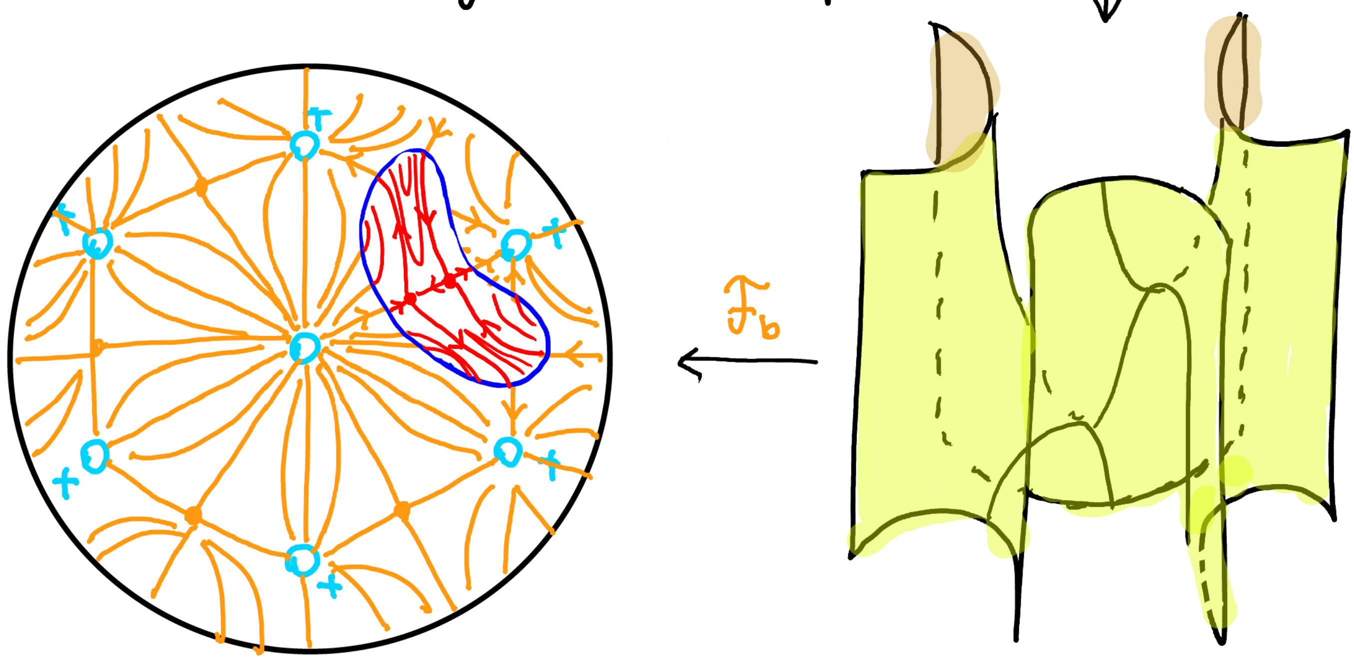


Reducing to the bigon

$N(D_s)$:



exchange order of saddle points



$\Rightarrow$ by induction ✓

HW What extra assumption do we need for the saddles?

We are done :

managed to remove all negative elliptic points $\Rightarrow$ $0 = e_- \leq h_-$

$$\Updownarrow \text{ seen}$$

$$sl(K) \leq -\chi(\Sigma)$$

Bennequin's Thm (re)started two fields:

(A) contact geometry

(B) braid foliations

(B) Braid foliations

Birman-Menasco:

- 3-braids: up to conjugation any link which is the closure of a 3-braid has a unique 3-braid representation except for
  - → unknot: $\sigma_1\sigma_2$, $\sigma_1^{-1}\sigma_2^{-1}$, $\sigma_1\sigma_2^{-1}$
  - → $(2,k)$ torus knots: $\sigma_1^t\sigma_2$, $\sigma_1^t\sigma_2^{-1}$
  - → closures of $\sigma_1^p\sigma_2^q\sigma_1^r\sigma_2^\delta$, $\sigma_1^p\sigma_2^\delta\sigma_1^r\sigma_2^q$

    where p,q,r distinct, $|\,|\geq 2$, $\delta = \pm 1$

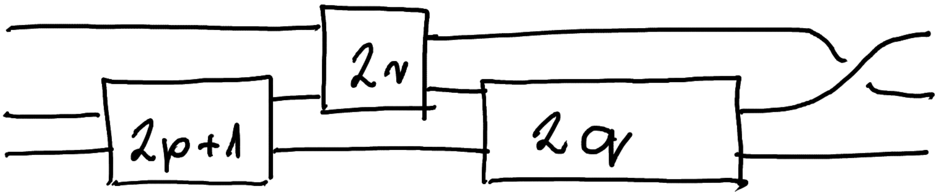
- Markov Thm w/o stabilisation: there are some extra braid index fixing braid isotopies s.t. any two braid representatives B, C of a link is connected

$$B = B_0 \to B_1 \to B_2 \to \dots \to B_k = C$$

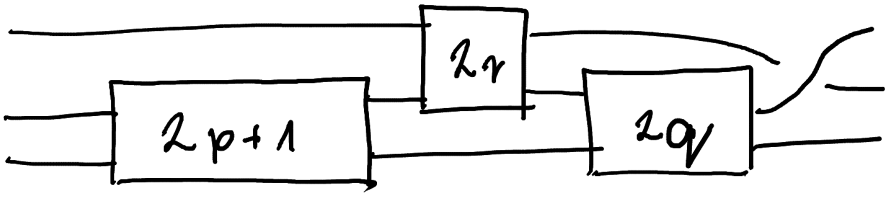
with monotone braid index

- Transversaly non-simple knots

the transverse knots represented by the braids



and



(where $p+1 \neq q \neq r \quad p,q,r > 1$)

are
- smoothly isotopic
- have same sl
- NOT transversaly isotopic

Rmk: Etnyre - Honda constructed such examples using convex surface theory (→ 3rd lecture)

(A) Contact geometry

contact structures
- overtwisted = $\exists$ transverse unknot w/ $sl(U)=1$
- tight = not overtwisted

Thm (Eliashberg '92): (M^3, ξ) contact

The following are equivalent:

- ξ is tight
- $sl(K) \leq -\chi(\Sigma)$ for all $\pitchfork$ K & $\partial\Sigma = K$
- $\{sl(K) : K \pitchfork\}$ is bounded above
- $\{sl(K) : K \pitchfork$ smoothly is isotopic to $\mathcal{K}\}$ is bounded above for all smooth knot-types $\mathcal{K}$

Thm (Eliashberg '92): $(\mathbb{R}^3, \xi)$ tight
$\Rightarrow$ ξ isotopic to ξ_{st}

Thm (Lutz-Martinet): Any planefield on any M closed is homotopic to an overtwisted contact structure.

Thm (Eliashberg '98): If two overtwisted contact structures are homotopic as planefields
$\Rightarrow$ isotopic.

Thm (Colin-Giroux-Honda, '09)
Only finitelly many homotopy classes are homotopic to tight contact structures

⋮