Fused braids and fused Hecke algebra

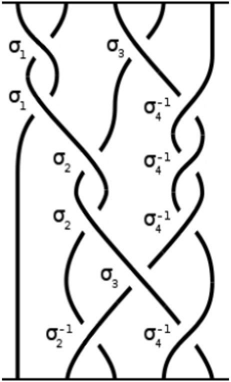
Winterbraids X, Pisa, February 2020

Section 1

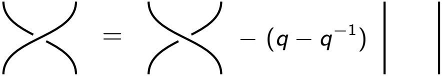
Previously in Winterbraids X...

Hecke algebra $H_n(q)$

Braid group :



Hecke algebra $H_n(q)$:



Algebraically :

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j| > 1,$$

$$\boxed{\sigma_i^2 = 1 + (q-q^{-1})\sigma_i}$$

$\rightsquigarrow$ HOMFLY-PT polynomial of a link.

Baxterisation

- **Yang–Baxter equation.** $R : \mathbb{C} \to \mathrm{End}(V \otimes V)$.

$$R_1(\alpha)R_2(\alpha\beta)R_1(\beta) = R_2(\beta)R_1(\alpha\beta)R_2(\alpha) \qquad \text{on } \underbrace{V \otimes \overbrace{V}^{}}_{R_1}\!\!\overbrace{\otimes V}^{R_2}$$

- If we set $R_i(\alpha) := \sigma_i + (q - q^{-1})\dfrac{1}{\alpha - 1}$ then

$$R_1(\alpha)R_2(\alpha\beta)R_1(\beta) = R_2(\beta)R_1(\alpha\beta)R_2(\alpha) \qquad \text{in } H_n(q)$$

- For any V there is a (local) representation :

$$H_n(q) \hookrightarrow \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$$

$\rightsquigarrow$ Solutions of YB.

Quantum groups and Schur–Weyl duality

- Say $\dim(V) = D$:

$$H_n(q) \hookrightarrow \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$$

Quantum groups and Schur–Weyl duality

- Say $\dim(V) = D$:

$$H_n(q) \hookrightarrow \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \hookleftarrow U_q(sl_D)$$

Quantum groups and Schur–Weyl duality

- Say $\dim(V) = D$:

$$H_n(q) \hookrightarrow \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} \hookleftarrow U_q(sl_D)$$

- From the point of view of representation theory :

Theorem (Schur–Weyl duality)

- *The image of $H_n(q)$ is the centraliser of the action of $U_q(sl_D)$ ($\forall D$)*

Example ($D = 2$, V is the spin $1/2$ representation of $U_q(sl_2)$) :

- the image of $H_n(q)$ is the *Temperley–Lieb algebra* (Jones pol.) ;

Summary

The Hecke algebra $H_n(q)$:

- Quotient of braid group algebra $\rightsquigarrow$ Knots and links invariants
- Explicit Baxterisation formula $\rightsquigarrow$ matrix solutions of YB on $V^{\otimes n}$
- Centraliser of $U_q(sl_D)$ on $V^{\otimes n}$

where V is the vector representation of $U_q(sl_D)$.

If $D = 2$ then V is the spin $1/2$ representation of $U_q(sl_2)$.

Goal : After applying fusion

A new algebra $H_{k,n}(q)$ $(\forall k)$

- Quotient of braid group algebra $\rightsquigarrow$ Knots and links invariants ?
- Explicit Baxterisation formula $\rightsquigarrow$ matrix solutions of YB on $W^{\otimes n}$
- Centraliser of $U_q(sl_D)$ on $W^{\otimes n}$

where $W = S^k(V)$ is the k-th symmetric power (for $U_q(sl_D)$).

If $D = 2$ then W is the spin $k/2$ representation of $U_q(sl_2)$.

Section 2

What is “fusion”?

Vector spaces :

$V \otimes V$

Solutions of YB eq. :

Vector spaces :

$V \otimes V$

$\downarrow$ (generic fusion)

$V^{\otimes k} \otimes V^{\otimes k}$

Solutions of YB eq. :

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$V^{\otimes k} \otimes V^{\otimes k} = S^k V \otimes S^k V \oplus \dots$

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$\downarrow$ (projection)

$S^k V \otimes S^k V$

Solutions of YB eq. :

- Denote $Proj \ : \ V^{\otimes k} \otimes V^{\otimes k} \to S^k V \otimes S^k V$.

Vector spaces :	Solutions of YB eq. :
$V \otimes V$	basic solution $R(u)$ on $V \otimes V$
$\downarrow$ (generic fusion)	
$V^{\otimes k} \otimes V^{\otimes k} = S^k V \otimes S^k V \oplus \ldots$	
$\downarrow$ (projection)	
$S^k V \otimes S^k V$	

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Solutions of YB eq. :

basic solution $R(u)$ on $V \otimes V$

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new solution $R^{(k)}(u)$ on $V^{\otimes k} \otimes V^{\otimes k}$

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Vector spaces :	Solutions of YB eq. :
$V \otimes V$	basic solution $R(u)$ on $V \otimes V$
$\downarrow$ (generic fusion)	$\downarrow$ (generic fusion)
$V^{\otimes k} \otimes V^{\otimes k} = S^k V \otimes S^k V \oplus \dots$	new solution $R^{(k)}(u)$ on $V^{\otimes k} \otimes V^{\otimes k}$
$\downarrow$ (projection)	$\downarrow$ (multiply by *Proj*)
$S^k V \otimes S^k V$	fused solution $R^{fus}(u)$ on $S^k V \otimes S^k V$

- Denote $Proj \ : \ V^{\otimes k} \otimes V^{\otimes k} \to S^k V \otimes S^k V$.

Vector spaces :

$V \otimes V$

$\downarrow$ (generic fusion)

$V^{\otimes k} \otimes V^{\otimes k} = S^k V \otimes S^k V \oplus \dots$

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$S^k V \otimes S^k V$

Solutions of YB eq. :

basic solution $R(u)$ on $V \otimes V$

$\downarrow$ (generic fusion)

new solution $R^{(k)}(u)$ on $V^{\otimes k} \otimes V^{\otimes k}$

$\downarrow$ (multiply by $Proj$)

fused solution $R^{fus}(u)$ on $S^k V \otimes S^k V$

- Denote $Proj \ : \ V^{\otimes k} \otimes V^{\otimes k} \to S^k V \otimes S^k V$.

Key point : $Proj \in End_{U_q(sl_D)}(\dots) \rightsquigarrow$ Hecke algebra .

- Step 1 (generic fusion).

Given arbitrary parameters $(c_1, \ldots, c_{2k})$, explicit formula for $R^{(k)}(u)$ and :

$$R^{(k)}(u) \text{ satisfies YB on } V^{\otimes k} \otimes V^{\otimes k}$$

- Step 1 (generic fusion).

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Note : if you are a *braid person*, you will think "cabling" and agree very quickly...

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• Step 2 (Projection with *Proj*).

Thm. : There is a specific choice of $(c_1, \ldots, c_{2k})$ such that

$$R^{(k)}(u) \text{ commutes with } Proj$$

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• Step 2 (Projection with *Proj*).

Thm. : There is a specific choice of $(c_1, \ldots, c_{2k})$ such that

$$R^{(k)}(u) \text{ commutes with } Proj$$

$\rightsquigarrow R^{fus}(u) :=$ the restriction on $S^k V \otimes S^k V$.

Matrices :

$R(u)$ on $V \otimes V$

$\downarrow$ (generic fusion)

$R^{(k)}(u)$ on $V^{\otimes k} \otimes V^{\otimes k}$

$\downarrow$ (projection)

$R^{fus}(u)$ on $S^k V \otimes S^k V$

Algebras :

Matrices :	Algebras :
$R(u)$ on $V \otimes V$	$\hookleftarrow$ Hecke algebra H_n
$\downarrow$ (generic fusion)	
$R^{(k)}(u)$ on $V^{\otimes k} \otimes V^{\otimes k}$	
$\downarrow$ (projection)	
$R^{fus}(u)$ on $S^k V \otimes S^k V$	

Matrices :

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Algebras :

$\leftarrow$ Hecke algebra H_n

$\leftarrow$ (bigger) Hecke algebra H_{kn}

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$R(u)$ on $V \otimes V$	$\hookleftarrow$ Hecke algebra H_n
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$R^{(k)}(u)$ on $V^{\otimes k} \otimes V^{\otimes k}$	$\hookleftarrow$ (bigger) Hecke algebra H_{kn}
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$R^{fus}(u)$ on $S^k V \otimes S^k V$	$\hookleftarrow$ *? ? fused Hecke algebra ? ?*

Matrices :	Algebras :
$R(u)$ on $V \otimes V$	$\hookleftarrow$ Hecke algebra H_n
$\downarrow$ (generic fusion)	
$R^{(k)}(u)$ on $V^{\otimes k} \otimes V^{\otimes k}$	$\hookleftarrow$ (bigger) Hecke algebra H_{kn}
$\downarrow$ (projection)	
$R^{fus}(u)$ on $S^k V \otimes S^k V$	$\hookleftarrow$?? *fused Hecke algebra* ??

- Answer : Fused Hecke algebra $=$ $\boxed{Proj \cdot H_{kn} \cdot Proj}$

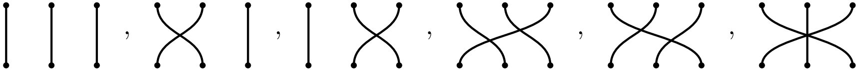
Section 3

Fused Hecke algebra

(joint work with Nicolas Crampé)

Symmetric group

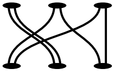
- Elements of S_n (example with $n = 3$) :

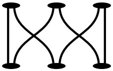


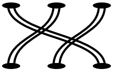
- Multiplication : concatenation + following the lines ($\sim$ composition).

Fused permutations ($q = 1$)

- Objects : Examples ($k = 2$ and $n = 3$) :



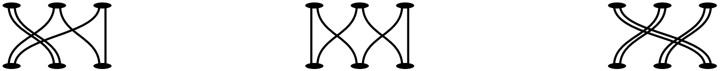




connecting dots with k lines starting from and arriving at each dot.

Fused permutations ($q = 1$)

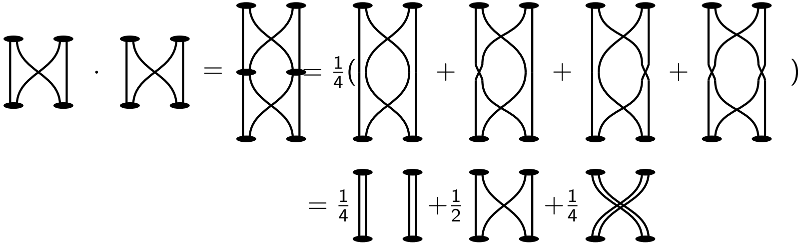
- Objects : Examples ($k = 2$ and $n = 3$) :



connecting dots with k lines starting from and arriving at each dot.

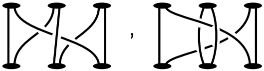
- multiplication : concatenation + following all possible paths.

Example :



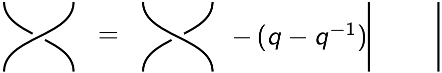
Fused Hecke algebra $H_{k,n}(q)$.

Deformation of the case $q = 1$.

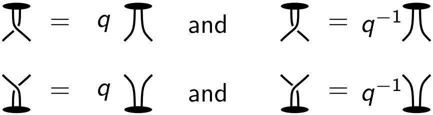
Example of objects (fused braids) : 

Homotopy + local relations :

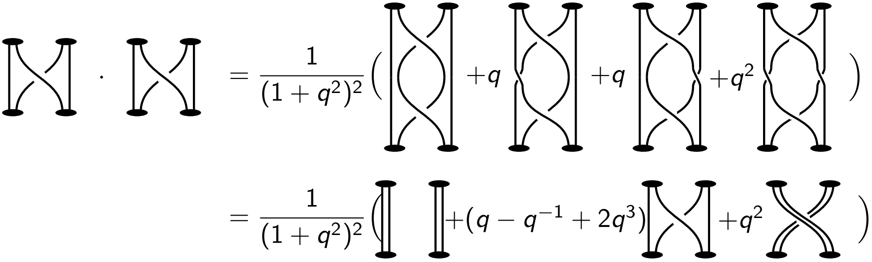
<u>The Hecke relation :</u>



<u>The idempotent relations :</u>



Multiplication : concatenation + following all paths with q-coefficients :



$\rightsquigarrow$ Facts : Family of algebras $H_{k,n}(q)$ forming a chain :

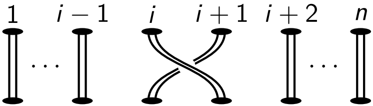
$$H_{k,1}(q) \subset H_{k,2}(q) \subset \dots \cdots \subset H_{k,n}(q) \subset H_{k,n+1}(q) \subset \dots$$

of finite-dimensional algebras, flat deformations of the case $q = 1$, with a basis of diagrams.

Braid group in $H_{k,n}(q)$

The shuffle elements :

$\Sigma_i :=$



$(k = 2)$

The elements Σ_i satisfy the braid relations :

$$\Sigma_i \Sigma_{i+1} \Sigma_i = \Sigma_{i+1} \Sigma_i \Sigma_{i+1}$$
$$\Sigma_i \Sigma_j = \Sigma_j \Sigma_i \qquad \text{if } |i-j| > 1.$$

$+$ a characteristic equation of order $k+1$.

Example $(k = 2)$: $\quad (\Sigma_i - q^4)(\Sigma_i + 1)(\Sigma_i - q^{-2}) = 0$

$\rightsquigarrow$ Finite-dimensional quotients of the braid group algebra inside $H_{k,n}(q)$.

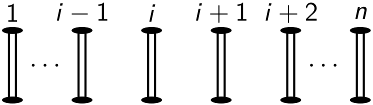
Theorem (Baxterisation formula)

The following satisfies the YB equation in $H_{k,n}(q)$:

$$R_i(\alpha) = \sum_{p=0}^{k} q^{k-p} \begin{bmatrix} k \\ p \end{bmatrix}_q^2 \frac{(1-q^{-2})\ldots\ldots(1-q^{-2(k-p)})}{(\alpha q^{-2(k-1)}-1)\ldots\ldots(\alpha q^{-2p}-1)} \Sigma_i^{(p)} ,$$

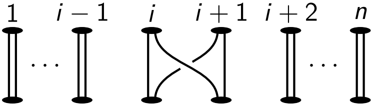
where $\Sigma_i^{(p)}$ are partial shuffle elements :

$\Sigma_i^{(0)} :=$



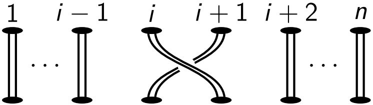
$(k = 2)$

$\Sigma_i^{(1)} :=$



$(k = 2)$

$\Sigma_i^{(2)} :=$



$(k = 2)$

Schur–Weyl duality

- $W = S^k(V)$ then :

$$H_{k,n}(q) \hookrightarrow \underbrace{W \otimes \cdots \otimes W}_{n \text{ times}}$$

$\rightsquigarrow$ Solutions of braid relation and YB on $W = S^k(V)$.

Schur–Weyl duality

$$\dim(V) = D$$

- $W = S^k(V)$ then :

$$H_{k,n}(q) \hookrightarrow \underbrace{W \otimes \cdots \otimes W}_{n \text{ times}} \hookleftarrow U_q(sl_D)$$

$\rightsquigarrow$ Solutions of braid relation and YB on $W = S^k(V)$.

- From the point of view of representation theory :

Theorem

- *The image of $H_{k,n}(q)$ is the centraliser of the action of $U_q(sl_D)$ $(\forall D)$.*

Kernel from representation theory

Step 1. Construct the Bratteli diagram.

Step 2. Understand which representations are in the kernel.

Theorem (Step 1. Bratteli diagrams)

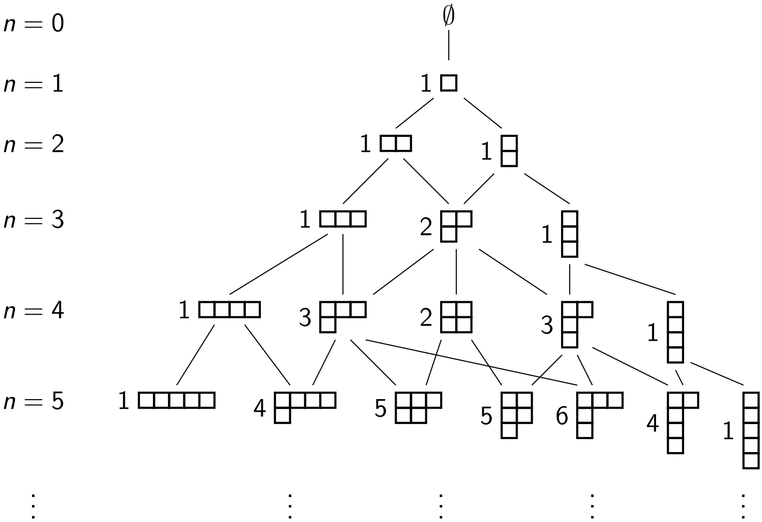
- *Irreducible reps of* $H_{k,n}(q)$ $\overset{1-1}{\longleftrightarrow}$ $\{\lambda \vdash kn \text{ with } \ell(\lambda) \leq n\}$
- *Branching rules :*

$$\mu \longrightarrow \lambda \quad \Leftrightarrow \quad \mu \subset \lambda \text{ and } \lambda/\mu \text{ contains at most one box per column}$$

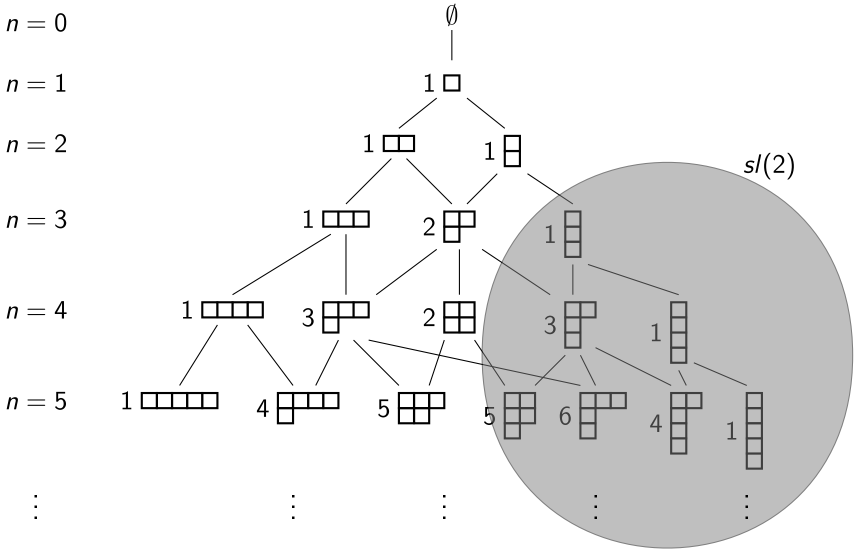
Theorem (Step 2. Kernel for $U_q(sl_D)$)

At level $D+1$*, kill all partitions with exactly* $D+1$ *lines and then all their descendants in the next levels.*

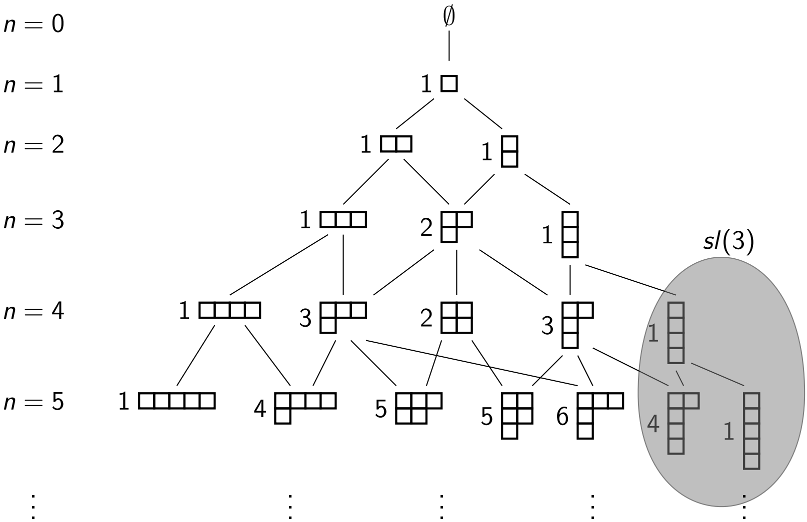
Bratelli diagram of $H_{k,n}(q)$ ($k = 1$)



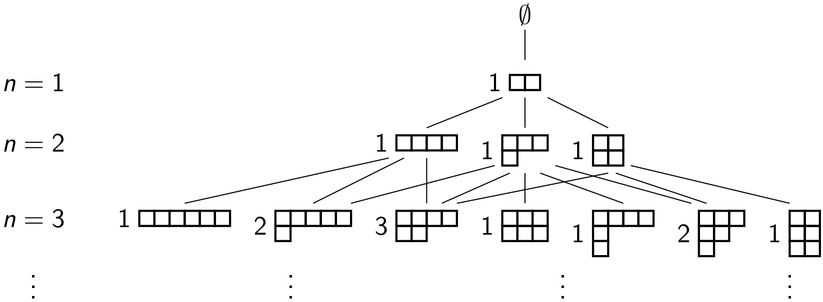
Centralisers for $k = 1$, $U_q(sl_2)$ (spin 1/2)

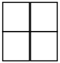
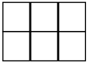


Centralisers for $k = 1$, $U_q(sl_3)$

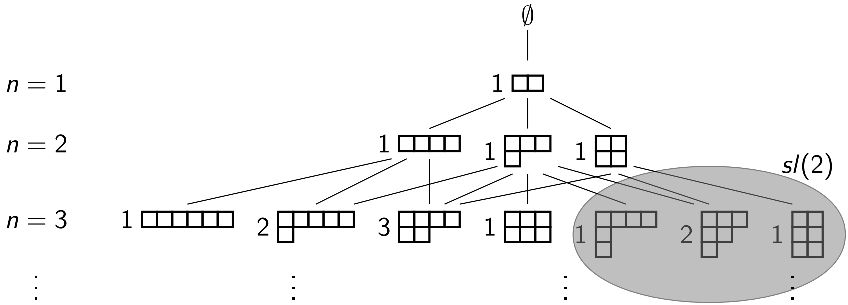


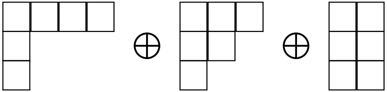
Representation theory of $H_{k,n}(q)$ $(k = 2)$



Note that there is no arrow from $\mu =$  to $\lambda =$  even if $\mu \subset \lambda$ since λ/μ contains two boxes in the same column.

Centralisers for $k = 2$, $U_q(sl_2)$ (Spin 1)



The kernel is generated at level 3 by 

Kernel algebraically

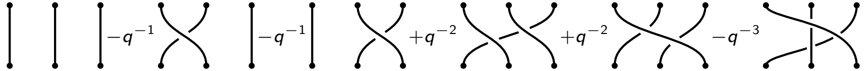
- Given k and $U_q(sl_D)$, the kernel is generated by the following element :

Theorem (Kernel)

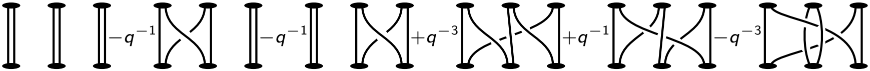
Start with the q-antisymmetriser on $D+1$ strands, and add $k-1$ vertical strands at each dot.

- Example $U_q(sl_2))$:

$k = 1$:



$k = 2$:



Conclusions

We defined algebras $H_{k,n}(q)$ on "fused braids" and :

- It contains a realisation of the **braid group**.
- There is an explicit **Baxterisation formula**.
- **Schur–Weyl duality for $W = S^k(V)$.**

  These algebras live above the **centralisers** of $U_q(sl_D)$:

$$\begin{array}{ccccccc} \dots \subset & H_{k,n}(q) & \subset & H_{k,n+1}(q) & \subset \dots \\ & \downarrow & & \downarrow & \\ \dots \subset & \mathrm{End}_{U_q(sl_D)}(W^{\otimes n}) & \subset & \mathrm{End}_{U_q(sl_D)}(W^{\otimes n+1}) & \subset \dots \end{array}$$

  $+$ Description of the kernels.