Ordered groups, knots, braids and hyperbolic 3-manifolds
Minicourse in Caen

Dale Rolfsen, UBC

February and March 2017

Outline

Lecture 1: Introduction to ordered groups

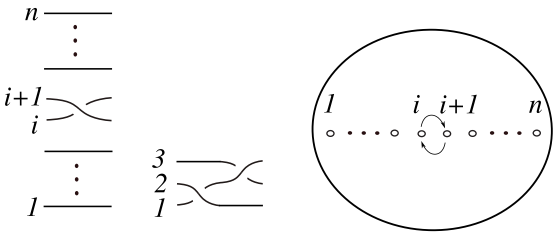
Lecture 2: Ordering knot groups; Fibred knots and surgery

Lecture 3: Braids, $Aut(F_n)$ and minimal volume hyperbolic 3-manifolds

This is joint work with Eiko Kin, Osaka University.

Braid groups

You are all familiar with the braid groups B_n, so I'll just review a few things to make my conventions clear.

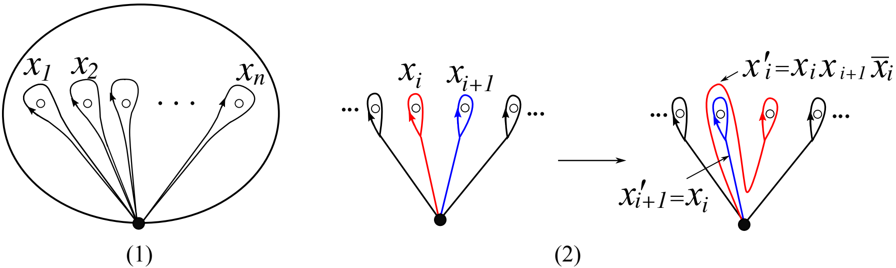


(1) (2) (3)

(1) pictures the braid $\sigma_i \in B_n$. (2) is the 3-braid $\sigma_1 \sigma_2^{-1}$ (3) shows the action of σ_i on the mapping class group of the n-punctured disk.

Braid groups

B_n acts on the fundamental group of the punctured disk, which is a free group F_n.



The Artin action of σ_i on $F_n \cong \langle x_1, \ldots, x_n \rangle$ is

$$x_i \to x_i x_{i+1} x_i^{-1} \quad x_{i+1} \to x_i \quad x_j \to x_j,\ j \neq i, i+1$$

Braid groups

The Artin representation is an injective homomorphism

$$B_n \to Aut(F_n).$$

We use the notation

$$x \to x^\beta$$

to denote the action of the braid β upon the group element $x \in F_n$ under this representation. Note that $x^{\beta\gamma} = (x^\beta)^\gamma$.
We say the braid $\beta \in B_n$ is order preserving if and only if there exists a bi-ordering $<$ of F_n such that

$$x < y \iff x^\beta < y^\beta$$

Braid groups

The central theme of today's talk is the interplay between braids and bi-orderings of free groups.

In particular, we study which braids are order-preserving. Many questions are still open.

This also has connections with orderability of certain link groups, and application to understanding minimal volume cusped hyperbolic 3-manifolds.

A discussion of ordering of free groups is in order....

Ordering free groups

One way to construct bi-orderings of F_n is via the lower central series

$$F_n = \gamma_0(F_n) \supset \gamma_1(F_n) \supset \gamma_2(F_n) \supset \cdots$$

defined inductively by $\gamma_{k+1}(F_n) = [\gamma_k(F_n), F_n]$.
These are all normal subgroups and have nice properties:

- $\gamma_k(F_n)/\gamma_{k+1}(F_n)$ is a finitely-generated free abelian group, that is, isomorphic with some $\mathbb{Z}^m$
- $\cap_{k=0}^{\infty}\gamma_k(F_n) = \{1\}$. That is, F_n is residually nilpotent.

Ordering free groups

To define an ordering on F_n it is enough to specify the positive elements.
For each $k \geq 0$ choose a bi-ordering $<_k$ of $\gamma_k(F_n)/\gamma_{k+1}(F_n)$.
So if $1 \neq x \in F_n$, let k be the largest integer such that $x \in \gamma_k(F_n)$ and say that

x is positive iff $1 <_k [x]$, where $[x]$ is its class in $\gamma_k(F_n)/\gamma_{k+1}(F_n)$.

It is routine to verify that this defines a bi-ordering of F_n. In fact, by varying the choices of $<_k$ one can define uncountably many different bi-orderings of F_n, if $n \geq 2$. Orderings constructed as we've described will be called LCS-type orderings.

Order-preserving braids

We begin with some relatively easy observations regarding order-preserving braids.

Proposition

The braid σ_i is not order-preserving.

To see this, recall that σ_i acts by $x_{i+1} \to x_i \to x_i x_{i+1} x_i^{-1}$.
If $<$ is a supposed invariant bi-ordering of F_n, we may assume w.l.o.g. that $x_i < x_{i+1}$.Then, by invariance, $x_i x_{i+1} x_i^{-1} < x_i$. Since bi-orderings are invariant under conjugation we conclude that $x_{i+1} < x_i$, a contradition.

Braid groups

Proposition

The full-twist n-braid $\Delta^2 = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ is order-preserving. In fact its action preserves every bi-ordering of F_n

That's because Δ^2 acts on F_n by conjugation, by $x_1x_2\ldots x_n$.
And every bi-ordering is invariant under conjugation.

Free group automorphisms

Note that any automorphism $\phi : F_n \to F_n$ takes each lower central subgroup into itself, so ϕ induces homomorphisms

$$\phi_k : \gamma_k(F_n)/\gamma_{k+1}(F_n) \to \gamma_k(F_n)/\gamma_{k+1}(F_n).$$

The homomorphism ϕ_0 is just the abelianization $\phi_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$.

Lemma

If $\phi_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$ is the identity mapping, so is every $\phi_k : \gamma_k(F_n)/\gamma_{k+1}(F_n) \to \gamma_k(F_n)/\gamma_{k+1}(F_n)$.

Theorem

If $\phi_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$ is the identity mapping, then $\phi : F_n \to F_n$ preserves every ordering of LCS - type.

Braid groups

Recall that a pure braid is one whose underlying permutation is the identity. The pure braids form a normal subgroup of B_n of index $n!$. Under the Artin representation, a pure braid sends each generator to some conjugate of itself. Such an automorphism abelianizes to the identity.

Corollary

If $\beta \in B_n$ is a pure braid, then β is order-preserving. In fact, β preserves every ordering of F_n of LCS-type.

Ordering extensions

We recall the HNN extension of a group K associated with an automorphism $\phi : K \to K$. If $k_1, \ldots, k_n$ are generators, we introduce a new symbol t and impose the relations

$$t^{-1} k_i t = k_i^{\phi}$$

Denote the resulting group $G = K \rtimes_{\phi} \mathbb{Z}$.

Proposition

Suppose K is bi-orderable and $\phi : K \to K$ is an automorphism. Then $G = K \rtimes_{\phi} \mathbb{Z}$ is bi-orderable if and only if there exists a bi-ordering of K which is preserved by ϕ.

Ordering extensions

An example of this is the fundamental group of a fibre bundle over the circle. If $h : X \to X$ is a homeomorphism of the space X, then the mapping torus is the space

$$\mathbb{T}_h := X \times [0,1]/(x,1) \sim (h(x),0).$$

There is a natural fibration $\mathbb{T}_h \to S^1$, with fibre X.
The fundamental group of the mapping torus is the extension

$$\pi_1(\mathbb{T}_h) \cong \pi_1(X) \rtimes_{h_*} \mathbb{Z}$$

where $h_* : \pi_1(X) \to \pi_1(X)$ is the 'homotopy monodromy.'

Braid groups

Back to the situation of braids, recall that a braid $\beta \in B_n$ acts on the punctured disk D_n. The mapping torus of this action is homeomorphic with the complement of the braided link $br(\beta) = \hat{\beta} \cup A$:

$$\mathbb{T}_\beta \cong S^3 \setminus br(\beta)$$

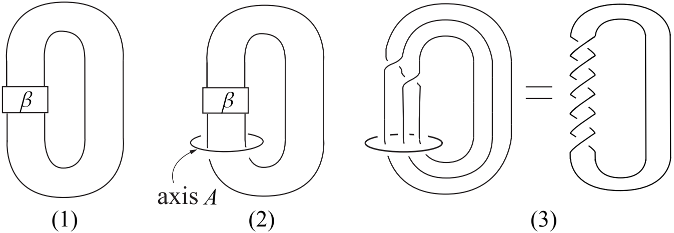


Figure: (1) Closure $\widehat{\beta}$. (2) $\mathrm{br}(\beta) = \widehat{\beta} \cup A$. (3) $\mathrm{br}(\sigma_1\sigma_2)$ is equivalent to the $(6, 2)$-torus link.

Braid groups

Proposition

For braid $\beta \in B_n$ the following are equivalent:

- β *is order preserving*
- *the fundamental group of* $\mathbb{T}_\beta$ *is bi-orderable*
- *the link group* $\pi_1(S^3 \setminus br(\beta))$ *is bi-orderable.*

Proposition

If a braid $\beta \in B_n$ is order-preserving, then so are all its conjugates.

Braid groups

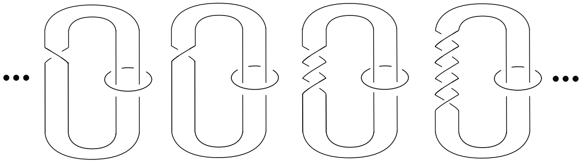


Figure: Links whose groups are not bi-orderable.

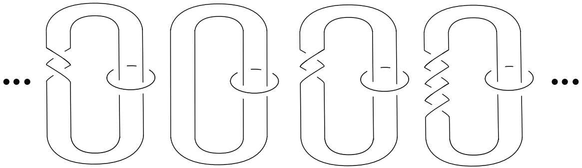


Figure: Links whose groups are bi-orderable.

Corollary

For every braid β some power β^k is order-preserving.

We call a braid $\beta \in B_n$ periodic if some power β^k lies in the centre of B_n. Recall that, for $n \geq 3$ the centre of B_n is infinite cyclic, generated by the full twist Δ_n^2.

Define $\delta_n = \sigma_1 \sigma_2 \cdots \sigma_{n-1}$. Noting that $\delta_n^n = \Delta_n^2$ we see that δ_n is periodic, being an n^{th} root of a full twist. There is also an $n-1$ root of a full twist, namely $\delta_n \sigma_1$.

Proposition

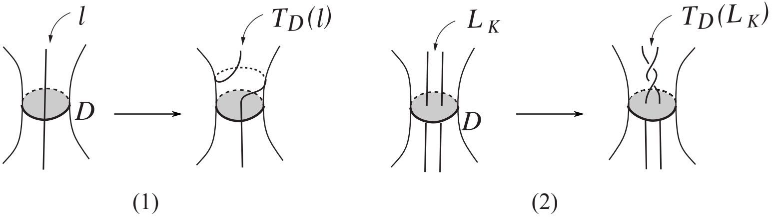
Every periodic braid is conjugate to a power of δ_n or $\delta_n \sigma_1$.

Periodic braids

Theorem

Let $\beta \in B_n$ be a periodic braid. If β is conjugate to $(\delta_n \sigma_1)^k$ then β is order-preserving. If β is conjugate to δ_n^k then β is NOT order-preserving, unless $k \equiv 0 \pmod{n}$.

Part of this theorem can be seen using a trick: a disk twist, which is a self-homeomorphism of the complement of an unknotted component of a link.



Braid groups

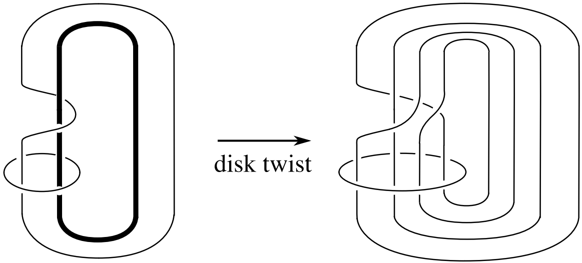


Figure: nth power of the disk twist converts the braided link of σ_1^2 to that of $\sigma_1\sigma_2\cdots\sigma_{n+1}\sigma_1$. ($n = 2$ in this case.)

Note that the link on the right and the link on the left have homeomorphic complements. The braid on the right is pure, therefore order-preserving, and so the complement of its braided link has bi-orderable group. We conclude that (in this picture) $\delta_4\sigma_1$ must be order-preserving.

Braid groups

Tensor product of braids.

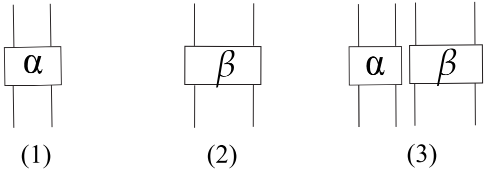


Figure: (1) $\alpha \in B_m$. (2) $\beta \in B_n$. (3) $\alpha \otimes \beta \in B_{m+n}$.

Proposition

The braid $\alpha \otimes \beta$ is order-preserving if and only if both α and β are order-preserving.

This follows from a recent theorem regarding the ordering of free products.

Braid groups

Theorem

Suppose $(G, <_G)$ and $(H, <_H)$ are bi-ordered groups. Then there is a bi-ordering of $G * H$ which extends the orderings of the factors and such that whenever $\phi : G \to G$ and $\psi : H \to H$ are order-preserving automorphisms, the ordering of $G * H$ is preserved by the automorphism $\phi * \psi : G * H \to G * H$.

Corollary

A braid $\beta \in B_m$ is order-preserving if and only if $\beta \otimes 1_n \in B_{m+n}$ is order-preserving.

Braid groups

Note that the order-preserving braids in B_2 are exactly the powers σ_1^k with k even. In other words, it is the subgroup of pure 2-braids.
For $n > 2$ the situation is different.

Proposition

For $n > 2$, the set of order-preserving braids is NOT a subgroup of B_n.

Consider $\alpha = \sigma_1\sigma_2\sigma_1$, which is (an extension of) the periodic braid $\delta_2\sigma_1 \in B_3$, hence order-preserving. Let $\beta = \sigma_1^{-2}$, a pure braid, so also order-preserving. But the product $\alpha\beta = \sigma_1\sigma_2\sigma_1^{-1}$ is not order-preserving, as it is conjugate to σ_2 which is not order-preserving.

Proposition

For $n > 2$, the set of order-preserving braids in B_n generates B_n.

Hyperbolic manifolds

We now turn attention to applications to understanding minimal volume hyperbolic manifolds, possibly with cusps.
The following results will be useful.

Theorem (Perron - R.)

Let $\phi : F_n \to F_n$ be an automorphism. If every eigenvalue of $\phi_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$ is real and positive, then there is a bi-ordering of F_n which is ϕ-invariant.

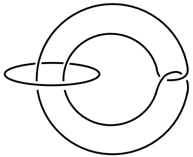
Theorem (Clay - R,)

If there exists a bi-ordering of F_n which is ϕ-invariant, then ϕ_{ab} has at least one real and positive eigenvalue.

Hyperbolic manifolds

Theorem (Gabai - Meyerhoff - Milley)

The (unique) minimal volume closed hyperbolic 3-manifold is the Weeks manifold, which can be obtained from the Whitehead link by $[5/2, 5/1]$ *surgery on the Whitehead link.*



Theorem (Calegari-Dunfield)

The fundamental group of the Weeks manifold is NOT left-orderable.

One-cusped hyperbolic manifolds

In the case of one cusp, there are two distinct examples.

Theorem (Cao-Meyerhoff)

A minimal volume one-cusped orientable hyperbolic 3-manifold is homeomorphic to either the complement of the figure-eight knot 4_1, or its sibling, which can be described as $5/1$ surgery on one component of the Whitehead link.

The following shows they can be distinguished by orderability properties of their fundamental groups.

Theorem

The figure-eight complement has bi-orderable fundamental group. The group of its sibling is NOT bi-orderable.

One-cusped hyperbolic manifolds

To see this, we note that both these manifolds can be realized as punctured torus bundles over S^1. In the case of the figure-eight complement, the (homology) monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has two positive eigenvalues $(3 \pm \sqrt{5})/2$. Thus the homotopy monodromy preserves an ordering of F_2, the fundamental group of the fibre, and therefore the mapping torus $S^3 \setminus 4_1$ has bi-orderable group.

The sibling has the monodromy $\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$. This has the two negative eigenvalues $(-3 \pm \sqrt{5})/2$. Therefore the homotopy monodromy cannot preserve a bi-order and so its mapping torus (the sibling) has NON-bi-orderable group.

Two-cusped hyperbolic manifolds

Theorem (Agol 2010)

A minimal volume orientable hyperbolic 3-manifold with 2 cusps is homeomorphic to either the Whitehead link complement or the $(-2, 3, 8)$*-pretzel link complement.*

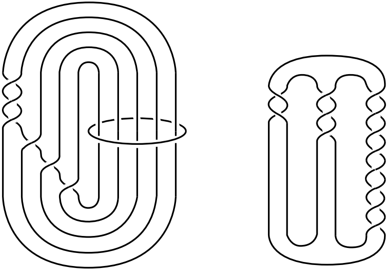


Figure: Two pictures of the $(-2, 3, 8)$-pretzel link. On the left, we may consider it the braided link $br(\delta_5\sigma_1^2)$

Two-cusped hyperbolic manifolds

Theorem

The fundamental group of the Whitehead link complement is bi-orderable. The group of the $(-2,3,8)$-pretzel link is NOT bi-orderable.

For the Whitehead link, whose complement fibres over S^1 with fibre a twice-punctured torus, one computes the homology monodromy $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, which has 1 as a triple eigenvalue. Therefore the homotopy monodromy preserves a bi-order of F_3, and so the group of the Whitehead link is bi-orderable.

Two-cusped hyperbolic manifolds

For the $(-2, 3, 8)$-pretzel, which is also $br(\delta_5\sigma_1^2)$, we conclude that its group cannot be bi-ordered by the observation:

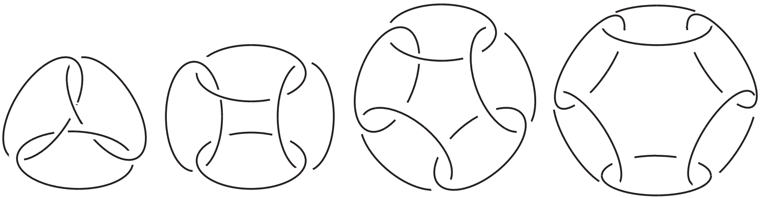
Proposition

For any $n \geq 3$ and positive integer k, the braid $\delta_n\sigma_1^{2k}$ is not order-preserving.

This can be proved by calculating the action of $\delta_n\sigma_1^{2k}$ on F_n, assuming it is order-preserving, and arriving at a contradiction.

More-cusped hyperbolic manifolds

Minimally-twisted chain links:



(1) (2) (3) (4)

Figure: (1) C_3. (2) C_4. (3) C_5. (4) C_6.

It is conjectured that for 3, 4, 5, 6 cusps, a minimal volume orientable hyperbolic manifold is homeomorphic with the complement of a "minimally twisted" chain link.

Four-cusped hyperbolic manifolds

Theorem (Yoshida)

A minimal volume orientable hyperbolic 3*-manifold with* 4 *cusps is homeomorphic to* $S^3 \setminus C_4$.

Theorem

$\pi_1(S^3 \setminus C_4)$ *is bi-orderable.*

Four-cusped hyperbolic manifolds

To prove this, we note that $S^3 \setminus C_4$ is homeomorphic (via a disk twist) to $br(\sigma_1^{-2}\sigma_2^2)$. The braid $\sigma_1^{-2}\sigma_2^2$ is order-preserving, as it is a pure braid.

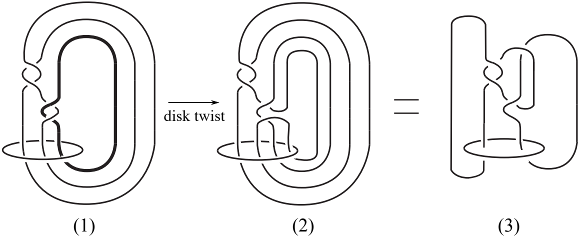


Figure: $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^2)$ is homeomorphic to $S^3 \setminus C_4$. (1) $\mathrm{br}(\sigma_1^{-2}\sigma_2^2)$. (2)(3) Links which are equivalent to C_4.

Five-cusped hyperbolic manifolds

For five cusps, the complement of the minimally twisted 5-chain is conjectured to be minimal among 5-cusped orientable hyperbolic manifolds.

Theorem

$\pi_1(S^3 \setminus C_5)$ *is bi-orderable.*

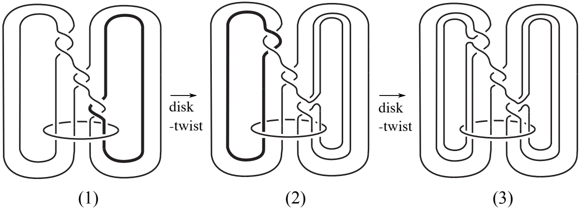


Figure: $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$ is homeomorphic to $S^3 \setminus C_5$. (1) $\mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$. (3) Link which is equivalent to C_5.

Six-cusped hyperbolic manifolds

Similarly we can show

Theorem

$\pi_1(S^3 \setminus C_6)$ *is bi-orderable.*

$S^3 \setminus C_6$ is conjectured to be minimal among 6-cusped examples.

Three-cusped hyperbolic manifolds

Similarly, $S^3 \setminus C_3$, a.k.a. the "magic manifold," is conjectured to be minimal among 3-cusped orientable hyperbolic 3-manifolds.
We do not know if its fundamental group is bi-orderable. One can realize $S^3 \setminus C_3$ as $br(\sigma_1^2\sigma_2^{-1})$. So we'll conclude with an open question.

Question: Is $\sigma_1^2\sigma_2^{-1} \in B_3$ order-preserving?