The mysterious geometry of Artin groups
Talk 2: On the edge

Jon McCammond

UC Santa Barbara

Caen
1 Mar 2016

Etratat, France

The big picture and the long view

Artin groups are defined by simple presentations, are closely related to Coxeter groups and have been studied since the 70s.

Remark (Algorithms)

Every Artin group that is understood and every part of an Artin group that is understood has very good algorithmic properties. The natural conjecture is that **all** Artin groups are well-behaved.

In the 1990s I heard Ruth Charney survey what was known and unknown about Artin groups and I remember being amazed at the vast amount that was **not** known about these groups, particularly since so much is known about Coxeter groups.

This talk is mostly a **lack-of-progress** report.

Two conventions for diagrams

Recall the two conventions for encoding Artin presentations.

m	Classical	Modern
2	no edge $\bullet \qquad \bullet$	no label $\bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet$
3	no label $\bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet$	label $\bullet\overset{3}{-\!\!-\!\!-\!\!-}\bullet$
> 3	label $\bullet\overset{m}{-\!\!-\!\!-\!\!-}\bullet$	label $\bullet\overset{m}{-\!\!-\!\!-\!\!-}\bullet$
∞	label $\bullet\overset{\infty}{-\!\!-\!\!-\!\!-}\bullet$	no edge $\bullet \qquad \bullet$

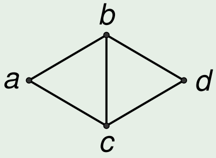
	Classical	Modern
disconnected	direct product	free product
no labels	small-type	right-angled

A small example

Example

A simple graph Γ and its small-type Artin group $\textsc{Art}(\Gamma)$:

$\Gamma =$



$$\textsc{Art}(\Gamma) = \left\langle a, b, c, d \;\middle|\; \begin{array}{ccc} aba = bab, & ad = da, & bdb = dbd \\ aca = cac, & bcb = cbc, & cdc = dcd \end{array} \right\rangle$$

Question (What do we know about this group?)

Does it have torsion? or a nontrivial center? How fast can we solve the word problem? What are its finiteness properties?

The basic conjectures

In a relatively recent survey article Eddy Godelle and Luis Paris highlight four basic conjectures about irreducible Artin groups:

Conjectures

A) *Every Artin group is torsion-free*
B) *Every non-spherical Artin group has trivial center*
C) *Every Artin group has a solvable word problem*
D) *Artin groups satisfy the* $K(\pi, 1)$ *conjecture*

Remark (Our small example is only partly understood)

Some conjectures are known for this example (Charney). For slightly more complicated graphs, all four conjectures are open.

Properties of general Coxeter groups

We have a much better understanding of Coxeter groups.

Theorem

Every Coxeter group

- *is defined by a simple presentation,*
- *has a faithful linear representation,*
- *is automatic,*
- *is a* $\mathrm{CAT}(0)$ *group.*

Coxeter groups act geometrically on a simply-connected non-positively curved piecewise euclidean cell complex known as the Davis complex with Moussong's metric. They fit into many of the powerful theories of geometric group theory and are algorithmically very nice.

From Coxeter groups to Artin groups

Definition (Braid group of an action)

To find the braid group of a group G acting on a space X (1) remove the points with non-trivial stabilizers, (2) quotient by the resulting free G-action and (3) take the fundamental group of the quotient.

Remark (Artin group origins)

Artin groups arise as the braid groups of Coxeter groups acting on the interior of their complexified Tits cones.

The $K(\pi, 1)$ conjecture refers to this space. For the details see Luis Paris' recent survey.

Parabolic subgroups

Definition (Parabolic subgroups)

A parabolic subgroup of an Artin group is the subgroup generated by a subset of the standard generating set. It is the image of an Artin group defined by restricting to this subset.

Remark (Injectivity)

If A is a subspace of X and there is a retraction from X to A, then $\pi_1(A)$ injects into $\pi_1(X)$. This does not require either fundamental group to have a decidable word problem.

Theorem (Parabolics are convex Artin groups)

Using a retraction Harm van der Lek proved that parabolic subgroups are Artin groups and Ruth Charney and Luis Paris recently showed that these parabolic subgroups are convex.

Local properties

Remark (Amalgamated free products)

Because parabolics inject, an Artin group with an infinity is an amalgamated free product of simpler Artin groups.

Concretely, if there is no relation between a and b in generating set S, then it is an amalgamated free product of the parabolics generated by $S \smallsetminus \{a\}$ and $S \smallsetminus \{b\}$ amalgamated along the parabolic generated by $S \smallsetminus \{a, b\}$.

Definition (Local properties)

Let $\mathcal{A}$ be a collection of Artin groups. We say an Artin group is locally in $\mathcal{A}$ if every (irreducible) parabolic subgroup with no infinities belongs to $\mathcal{A}$. $loc(\mathcal{A})$ is the collection of Artin groups that are locally in $\mathcal{A}$. If $\mathcal{A}$ is “understood”, so is $loc(\mathcal{A})$.

Right-angled Artin groups

Definition (Right-angled Artin groups)

The collection $\mathcal{Z}$ of free abelian groups is set the Artin groups where all generators commute. An Artin group is right-angled if every relation is a commutation, i.e. every m is either 2 or ∞. Note that right-angled Artin groups are the class $loc(\mathcal{Z})$ of locally abelian Artin groups.

Remark (Properties and uses)

Every right-angled Artin group is the fundamental group of a non-positively curved cube complex and this solves the four basic conjectures for $loc(\mathcal{Z})$. Right-angled Artin groups play a prominent role in the Bestvina-Brady examples and in the recent work of Agol, Wise and their coauthors.

Spherical Artin groups

Remark (Spherical Artin groups)

The systematic study of Artin groups starts in 1972 with the pair of adjacent articles in the *Inventiones* by Deligne and by Brieskorn and Saito. They establish all four basic conjectures for the class $\mathcal{S}$ of spherical Artin groups.

Remark (Garside structures)

The Deligne argument is geometric and only applies to the ones which are crystallographic. The Brieskorn-Saito argument is more algebraic and applies to all spherical Artin groups. Dehornoy and Paris axiomatized these algebraic arguments to define Garside structures.

The class $loc(\mathcal{S})$ of locally spherical Artin groups is called FC-type in the literature.

Euclidean Artin groups

A few years ago Robert Sulway and I proved the following:

Theorem (Euclidean Artin groups)

Every irreducible euclidean Artin group is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space.

The proof uses the infinite dual presentations and an infinite presentation version of Garside theory.

Remark (Locally euclidean)

Let $\mathcal{E}$ denote the class of euclidean Artin groups in the broad sense that includes the spherical ones. The class $loc(\mathcal{E})$ of locally euclidean Artin groups is a natural extension of the Artin groups of FC-type and it is also “understood”.

Large-type and 2-dimensional Artin groups

Remark (Large-type Artin groups)

An Artin group is large-type if its presentation has no commuting relations, i.e. every m is at least 3. This class $\mathcal{L}$ can be understood using a variant of small cancellation theory.

Remark (2-dimensional Artin groups)

An Artin group is 2-dimensional if every 3-generator parabolic is not spherical and this includes all large-type Artin groups. Chermak solved the word problem for these $\mathcal{D}$ groups.

Remark (Charney's extension)

Charney proved the $K(\pi, 1)$ conjecture for an extension of the class of 2-dimensional Artin groups. It is unclear (to me) whether the word problem is solvable for this class $\mathcal{C}$ of groups.

Summary

Recall that $\mathcal{Z}$ is abelian, *loc*$(\mathcal{Z})$ is right-angled, $\mathcal{S}$ is spherical, *loc*$(\mathcal{S})$ is FC-type, $\mathcal{E}$ is euclidean (in the broad sense) and the class *loc*$(\mathcal{E})$ is new. There are natural inclusions among them.

$$\begin{array}{ccccc} loc(\mathcal{Z}) & \hookrightarrow & loc(\mathcal{S}) & \hookrightarrow & loc(\mathcal{E}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{Z} & \hookrightarrow & \mathcal{S} & \hookrightarrow & \mathcal{E} \end{array}$$

On the other hand we have large-type, 2-dimensional and Charney's extension which are already locally closed.

$$\mathcal{L} \hookrightarrow 2\mathcal{D} \hookrightarrow \mathcal{C}$$

The word problem is solved for groups in *loc*$(\mathcal{E} \cup 2\mathcal{D})$. The $K(\pi, 1)$ conjecture is solved for groups in *loc*$(\mathcal{S}^+ \cup \mathcal{C})$.

Artin monoids

Artin monoids are much easier to work with.

Remark (Artin monoid word problem)

The word problem for an Artin **monoid** is trivially solvable because the all of the relations preserve length and there are only finitely many words of a fixed length.

This is not practical, but it means that they can at least be easily investigated, and there are good algorithms.

Remark (Does the Artin monoid embed in its group?)

It is not immediately obvious, however, whether or not the natural map from an Artin monoid to the corresponding Artin group with the same presentation is injective.

Artin monoids inject

This was only shown in 2002 and depends on the linearity of the braid groups.

Theorem (Braid groups are linear)

Daan Krammer and Stephen Bigelow proved that braid groups are linear around 2000.

Remark (Extensions)

Digne and Cohen-Wales extended Krammer's algebraic proof to show that every spherical Artin group is linear. Luis Paris extended this further to arbitrary Artin groups, but in the general case, it only proves the following.

Theorem (Artin monoids inject)

Every Artin monoid injects into the corresponding Artin group.

Tits' conjecture

Remark (Tits' conjecture)

Jacques Tits conjectured that the squares of the standard generators of an Artin group generate a subgroup that is a right-angled Artin group with a^2 and b^2 commuting if and only if a and b commute.

This subgroup is an image of a right-angled Artin group.

Theorem (Tits' conjecture is true)

John Crisp and Luis Paris proved Tits' conjecture in 2001 by finding a mapping class group representation of the Artin group in which no collapse of the right-angled Artin group occurs.

Cubulating Artin groups?

Wise, Agol and their coauthors have achieved spectacular successes by reducing questions about 3-manifolds to questions about right-angled Artin groups. The process they use is called cubulation.

Question (Cubulations)

Can all Artin groups be cubulated? i.e. does every Artin group virtually embed into a right-angled Artin group in a nice way?

No, not even all 2-dimensional Artin groups can be cubulated! Even $\textsc{Braid}_4$ cannot be (virtually) cubulated (Huang-Jankiewicz-Przytycki, Haettel).

The end of known world

And that concludes our quick tour of the known positive results about the word problem for Artin groups.

Remark (Missing results)

There are many more results that have been shown about those classes of Artin groups where the word problem or the $K(\pi, 1)$ conjecture has been solved. I am passing over those in silence to highlight the vast void at the center of the field.

In the final few slides I would like to highlight how much remains to be established. There is so much that we still don't know.

Smaller labels

Remark (Monoid injectivity)

John Crisp created a general method that can be used to prove that one Artin monoid injects into another.

Theorem (Small-type)

Luis Paris uses John Crisp's method to establishes that every Artin monoid injects into an Artin monoid of small type.

It would be natural to conjecture that these injections on the monoid level extend to the group level.

Remark (Conjectually universal Artin groups)

The remainder of the talk focuses on small-type Artin groups.

The small-type Artin groups we understand

Let Γ be a connected simple graph and let $G = \text{ART}(\Gamma)$ be the corresponding irreducible small-type Artin group.

Remark

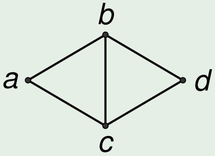
The only cases where we know how to solve the word problem for G are where Γ is:

1. a complete graph (b/c G is large-type),
2. an ADE Dynkin diagram (b/c G is spherical) and
3. an extended ADE Dynkin diagram (b/c G is euclidean)

And that is it! In other words, every small-type Artin group defined by any connected graph that is not complete, not a Dynkin diagram and not an extended Dynkin diagram has a word problem that we do not know how to solve!

A small example revisited

Example

$\Gamma =$ 

$$\textsc{Art}(\Gamma) = \left\langle a, b, c, d \;\middle|\; \begin{array}{ccc} aba = bab, & ad = da, & bdb = dbd \\ aca = cac, & bcb = cbc, & cdc = dcd \end{array} \right\rangle$$

This is a border case since it is contained in Charney's extension. The $K(\pi, 1)$ conjecture is known but we do not know how to solve the word problem. For almost all small-type Artin groups our ignorance is more extreme.

Bipartite graphs

Definition (Bipartite and small-type)

Let $K_{m,n}$ be the complete bipartite graph with m vertices on one side and n vertices on the other.

$$\mathrm{ART}(K_{m,n}) = \left\langle \begin{array}{c} a_1, a_2, \ldots, a_m \\ b_1, b_2, \ldots, b_m \end{array} \middle| \begin{array}{ll} a_i a_{i'} = a_{i'} a_i, & a_i b_j a_i = b_j a_i b_j \\ b_j b_{j'} = b_{j'} b_j & \end{array} \right\rangle$$

Remark (Known bipartite Artin groups)

We only know how to solve the word problem for $\mathrm{ART}(K_{m,n})$ in the 5 cases where $mn \leq 4$. It is spherical for $mn < 4$ and euclidean for $mn = 4$: $K_{1,1} = A_2$, $K_{1,2} = A_3$, $K_{1,3} = D_4$, $K_{1,4} = \widetilde{D}_4$ and $K_{2,2} = \widetilde{A}_3$. The rest are hyperbolic type and not understood.

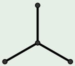
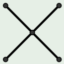
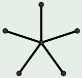
Stars/Claws

Definition

A star or claw is a connected graph $K_{1,n}$ in which all edges have a common endpoint.

For stars/claws progress has been **extremely** slow.

Example

				
$\textsc{Braid}_3$	$\textsc{Braid}_4$	$\textsc{Art}(D_4)$	$\textsc{Art}(\widetilde{D}_4)$	$\textsc{Art}(K_{1,5})$
Dehn	Artin	Br-Sa/Deligne	M-Sulway	???
1910s	1940s	1970s	2010s	???