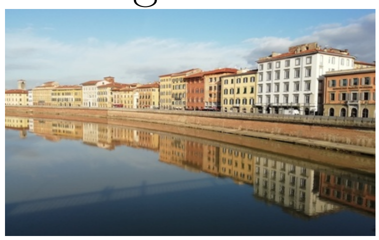
Invariants of links and 3-manifolds that count graph configurations.



Winter Braids X, Pisa, February 17th – 21th 2020

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Abstract

We present ways of counting configurations of uni-trivalent Feynman graphs in 3-manifolds in order to produce invariants of these 3-manifolds and of their links, following Gauss, Witten, Bar-Natan, Kontsevich and others. We first review the construction of the simplest invariants that can be obtained in our setting. These invariants are the linking number and the Casson invariant of integer homology 3-spheres. Next we see how the involved ingredients, which may be explicitly described using gradient flows of Morse functions, allow us to define a functor on the category of framed tangles in rational homology cylinders. Finally, we show some properties of our functor, which generalizes both a universal Vassiliev invariant for links in the ambient space and a universal finite type invariant of rational homology 3-spheres.

This is a preliminary version of the notes of a series of lectures given in Pisa in February 2020 for Winter Braids. It contains all what has been said during the lectures, and more. We refer to the book [Les20], where the above functor has been constructed and where all its mentioned properties are carefully proved, for more details. These notes may also be used as an introduction to [Les20] or as a reading guide for this book.

I warmly thank the organisers of this great session of Winter Braids.

Comments are welcome !

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MSC: 57M27 57N10 55R80 57R20

Contents

1 On the linking number and the Theta invariant

The modern powerful invariants of links and 3–manifolds that are studied in these series of lectures can be thought of as generalizations of the linking number. In this section, we warm up with several ways of defining this classical basic invariant. This allows us to introduce conventions and methods that will be useful througout these notes.

1.1 The linking number as a degree

Let S^1 denote the unit circle of the complex plane $\mathbb{C}$.

$$S^1 = \{z; z \in \mathbb{C}, |z| = 1\}.$$

Consider a C^∞ embedding

$$J \sqcup K \colon S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$$

of the disjoint union $S^1 \sqcup S^1$ of two circles into the ambient space $\mathbb{R}^3$ as the one pictured in Figure 1. Such an embedding represents a 2–*component link.*

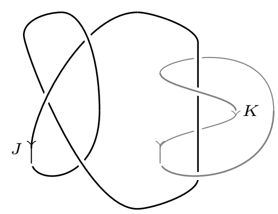
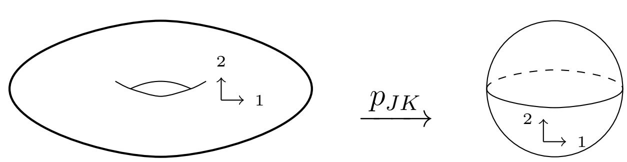


Figure 1: A 2–component link in $\mathbb{R}^3$

It induces the *Gauss map*

$$\begin{array}{cccc} p_{JK}\colon & S^1 \times S^1 & \hookrightarrow & S^2 \\ & (w,z) & \mapsto & \frac{1}{\|K(z)-J(w)\|}(K(z)-J(w)) \end{array}$$



Definition 1.1 The *Gauss linking number* $lk_G(J,K)$ of the disjoint *knots* $J(S^1)$ and $K(S^1)$, which are simply denoted by J and K, is the degree of the Gauss map p_{JK}.

There are several (fortunately equivalent) definitions of the degree for a continuous map between two closed (connected, compact, without boundary) oriented manifolds. Let us quickly recall our favorite one for these lectures.

Definition 1.2 A point y is a *regular value* of a smooth map $p\colon M \to N$ between two smooth manifolds M and N, if $y \in N$ and for any $x \in p^{-1}(y)$ the tangent map $T_x p$ at x is surjective[1]

An *orientation* of a real vector space V of positive dimension is a basis of V up to a change of basis with positive determinant. When $V = \{0\}$, an orientation of V is an element of $\{-1, 1\}$. An *orientation* of a smooth n–manifold is an orientation of its tangent space at each point, defined in a continuous way. (A local diffeomorphism h of $\mathbb{R}^n$ is orientation-preserving at x if and only if the Jacobian determinant of its derivative $T_x h$ is positive. If the transition maps $\phi_j \circ \phi_i^{-1}$ of an *atlas* $(\phi_i)_{i \in I}$ of a manifold M are orientation-preserving (at every point) for $\{i, j\} \subset I$, then the manifold M is *oriented* by this atlas.) Unless otherwise mentioned, manifolds are oriented in these notes.

When M and N are oriented, when M is compact and when the dimension of M coincides with the dimension of N, the *differential degree* of p at a regular value y of N is the (finite) sum running over the $x \in p^{-1}(y)$ of the signs of the determinants of $T_x p$. In this case, this differential degree can be extended to a continuous function $\deg(p)$ from the complement $N \setminus p(\partial M)$ of the image of the boundary ∂M of M to $\mathbb{Z}$. In particular, when M has no boundary, and when N is connected, the mentioned function is constant, its value is the *degree* of p. See [Mil97, Chapter 5].

The Gauss linking number $lk_G(J, K)$ can be computed from a link diagram like the one of Figure 1 as follows. It is the differential degree of p_{JK} at the vector Y that points towards us. The set $p_{JK}^{-1}(Y)$ is made of the pairs of points (w, z) where the projections of $J(w)$ and $K(z)$ coincide, and $J(w)$ is under $K(z)$. They correspond to the *crossings* ${}^{J}\nwarrow\!\!\nearrow{}^{K}$ and ${}^{K}\nwarrow\!\!\nearrow{}^{J}$ of the diagram.

In a diagram, a crossing is *positive* if we turn counterclockwise from the arrow at the end of the upper strand towards the arrow of the end of the lower strand like $\overset{\curvearrowleft}{\nwarrow\!\!\nearrow}$. Otherwise, it is *negative* like $\nwarrow\!\!\nearrow$.

For the positive crossing ${}^{J}\nwarrow\!\!\nearrow{}^{K}$, moving $J(w)$ along J following the orientation of J, moves $p_{JK}(w, z)$ towards the South-East direction, while moving $K(z)$ along K following the orientation of K, moves $p_{JK}(w, z)$ towards the North-East direction, so that the local orientation induced by the image of p_{JK} around $Y \in S^2$ is $\left\langle\begin{smallmatrix} Tpdz \\ Tpdw \end{smallmatrix}\right.$, which is $\left\langle\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right.$. Therefore, the contribution of a positive crossing to the degree is 1. It is easy to deduce that the contribution of a negative crossing is (-1).

We have proved the following formula

$$\deg_Y(p_{JK}) = \sharp\, {}^{J}\nwarrow\!\!\nearrow{}^{K} - \sharp\, {}^{K}\nwarrow\!\!\nearrow{}^{J}$$

where $\sharp$ stands for the cardinality –here $\sharp\, {}^{J}\nwarrow\!\!\nearrow{}^{K}$ is the number of occurences of ${}^{J}\nwarrow\!\!\nearrow{}^{K}$ in the diagram– so that

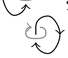
$$lk_G(J, K) = \sharp\, {}^{J}\nwarrow\!\!\nearrow{}^{K} - \sharp\, {}^{K}\nwarrow\!\!\nearrow{}^{J}.$$

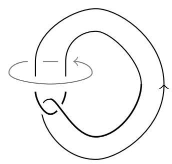
[1] According to the Morse-Sard theorem [Hir94, Chapter 3, Theorem 1.3, p. 69], the set of regular values of such a map is dense. (It is even *residual*, i.e. it contains the intersection of a countable family of dense open sets.) If M is compact, it is furthermore open.

Similarly, $\deg_{-Y}(p_{JK}) = \sharp\,{}^{K}{\nwarrow\!\!\!\!\!\nearrow}{}^{J} - \sharp\,{}^{J}{\nearrow\!\!\!\!\!\nwarrow}{}^{K}$ so that

$$lk_G(J,K) = \sharp\,{}^{K}{\nwarrow\!\!\!\!\!\nearrow}{}^{J} - \sharp\,{}^{J}{\nearrow\!\!\!\!\!\nwarrow}{}^{K} = \frac{1}{2}\left(\sharp\,{}^{J}{\nearrow\!\!\!\!\!\nwarrow}{}^{K} + \sharp\,{}^{K}{\nwarrow\!\!\!\!\!\nearrow}{}^{J} - \sharp\,{}^{K}{\nearrow\!\!\!\!\!\nwarrow}{}^{J} - \sharp\,{}^{J}{\nearrow\!\!\!\!\!\nwarrow}{}^{K}\right)$$

and $lk_G(J,K) = lk_G(K,J)$.

In the example of Figure 1, $lk_G(J,K) = 2$. Let us draw some further examples.
For the *positive Hopf link* J  K, $lk_G(J,K) = 1$.
For the *negative Hopf link* , $lk_G(J,K) = -1$.

For the *Whitehead link* , $lk_G(J,K) = 0$.

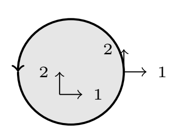
Since the differential degree of the Gauss map p_{JK} is constant on the set of regular values of p_{JK}, $lk_G(J,K) = \int_{S^1 \times S^1} p_{JK}^*(\omega_S)$ for any 2-form ω_S on S^2 such that $\int_{S^2} \omega_S = 1$.

Denote the standard area form of S^2 by $4\pi\omega_{S^2}$ so that ω_{S^2} is the homogeneous volume form of S^2 such that $\int_{S^2} \omega_{S^2} = 1$. In 1833, Gauss defined the linking number of J and K, as an integral [Gau77]. With modern notation, his definition reads

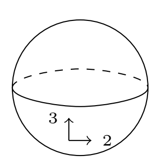
$$lk_G(J,K) = \int_{S^1 \times S^1} p_{JK}^*(\omega_{S^2}).$$

1.2 The linking number as an algebraic intersection

The boundary ∂M of an oriented manifold M is oriented by the *outward normal first* convention. If $x \in \partial M$ is not in a ridge, the outward normal to M at x followed by an oriented basis of $T_x\partial M$ induce the orientation of M. For example, the standard orientation of the disk in the plane induces the standard orientation of the circle, counterclockwise, as the following picture shows.



As another example, the sphere S^2 is oriented as the boundary of the ball B^3, which has the standard orientation induced by (Thumb, index finger (2), middle finger (3)) of the right hand.



The tangent bundle to an oriented submanifold A in a manifold M at a point x is denoted by T_xA. Two submanifolds A and B in a manifold M are *transverse*[2] if at each intersection point x, $T_xM = T_xA + T_xB$. If two transverse submanifolds A and B in a manifold M are of complementary dimensions (i.e. if the sum of their dimensions is the dimension of M), then the *sign of an intersection point* is $+1$ if $T_xM = T_xA \oplus T_xB$ as oriented vector spaces . Otherwise, the sign is -1. If A and B are compact and if A and B are of complementary dimensions in M, their *algebraic intersection* is the sum of the signs of the intersection points, it is denoted by $\langle A, B\rangle_M$.

For us, a *rational chain* (resp. *integral chain*) is a linear combination of (oriented) smooth manifolds with boundary, with coefficients in $\mathbb{Q}$ (resp.in $\mathbb{Z}$). Algebraic intersection bilinearly extends to pairs of chains.

When $\mathbb{K}$ is $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$ or $\mathbb{Q}$, a $\mathbb{K}$–S^3 or $\mathbb{K}$*–sphere* is a compact oriented 3-dimensional manifold[3] R with the same homology with coefficients in $\mathbb{K}$ as the standard unit sphere S^3 of $\mathbb{R}^4$. $\mathbb{Q}$-spheres (resp. $\mathbb{Z}$-spheres) are also called rational (resp. integer) homology 3-spheres. In these notes, we drop the 3 since the ambient dimension is always 3.

Any knot K in a $\mathbb{Q}$–sphere R bounds[4] an oriented *rational chain* in R. If R is a $\mathbb{Z}$-sphere, K bounds an embedded surface[5], which is called a *Seifert surface of the knot.*

The simplest definition of the linking number of two disjoint knots (represented by embeddings of S^1) in such a manifold is the following one.

Definition 1.3 The *linking number* $lk(J, K)$ of two disjoint knots J and K in a $\mathbb{Q}$-sphere R is the algebraic intersection $\langle J, \Sigma_K\rangle_R$ of J and a rational chain Σ_K bounded by K.

We will see that $lk_G(J, K) = lk(J, K)$ for 2-component links in $\mathbb{R}^3 \subset S^3$ in Lemma 1.15. See also [Les20, Proposition 2.8].

Let us now rephrase the definition of the Gauss linking number in ways which will generalize to 2–component links in a rational homology sphere R.

As in Subsection 1.1, consider a two-component link $J \sqcup K : S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$. This embedding induces an embedding

$$\begin{array}{rccc} J \times K\colon & S^1 \times S^1 & \hookrightarrow & (\mathbb{R}^3)^2 \setminus \mathrm{diag} \\ & (z_1, z_2) & \mapsto & (J(z_1), K(z_2)). \end{array}$$

Consider the following map

$$\begin{array}{rccc} p_{S^2}\colon & ((\mathbb{R}^3)^2 \setminus \mathrm{diag}) & \to & S^2 \\ & (x, y) & \mapsto & \frac{1}{\|y-x\|}(y - x). \end{array}$$

The Gauss map p_{JK} of Section 1.1 reads $p_{S^2} \circ (J \times K)$.

[2] As shown in [Hir94, Chapter 3 (Theorem 2.4 in particular)], transversality is a generic condition.

[3] Here, all manifolds are supposed to be smooth. Since any topological 3-manifold has a unique smooth structure (see [Kui99]), we do not specify "smooth" and we often only describe 3-manifolds up to homeomorphisms.

[4] This property characterizes $\mathbb{Q}$–spheres among closed oriented 3-manifolds.

[5] This property characterizes $\mathbb{Z}$–spheres among closed oriented 3-manifolds.

In particular, we can rewrite $lk_G(J,K)$ as another algebraic intersection, which will generalize to 2–component links in a rational homology sphere R. For a regular value $a \in S^2$ of p_{JK},

$$lk_G(J,K) = \deg_a p_{JK} = \langle (J \times K)(S^1 \times S^1), p_{S^2}^{-1}(a)\rangle_{(\mathbb{R}^3)^2 \setminus \mathrm{diag}}$$

where the preimages are oriented as follows. The *normal bundle* T_xM/T_xA to A in M at x is denoted by N_xA. It is oriented so that (a lift of an oriented basis of) N_xA followed by (an oriented basis of) T_xA induce the orientation of T_xM. The orientation of $N_x(A)$ is a *coorientation* of A at x. The regular preimage of a submanifold under a map f is oriented so that f preserves the coorientations.

For any 2-form ω_S on S^2 such that $\int_{S^2} \omega_S = 1$, we can also use the closed 2-form $p_{S^2}^*(\omega_S)$ of $(\mathbb{R}^3)^2 \setminus \mathrm{diag}$ to write

$$lk_G(J,K) = \int_{S^1 \times S^1} p_{JK}^*(\omega_S) = \int_{(J\times K)(S^1 \times S^1)} p_{S^2}^*(\omega_S)$$

.

The closure of $p_{S^2}^{-1}(a)$ in a compactification $C_2(S^3)$ of the 2–point configuration space $\left(\check{C}_2(S^3) = (\mathbb{R}^3)^2 \setminus \mathrm{diag}\right)$ is our first example of *propagating chain* or *propagator*. The closed 2-form $p_{S^2}^*(\omega_S)$ extends to $C_2(S^3)$ as an example of *propagating form* or *propagator*. Propagators are central ingredients in the construction of more general invariants of tangles in $\mathbb{Q}$–spheres that is presented below.

1.3 Propagators

Let us first introduce the compact 2–point configuration spaces where propagators live. Their constructions use the following differential blow-ups.

Definition 1.4 Recall that the *unit normal bundle* of a submanifold C in a smooth manifold A is the fiber bundle whose fiber over $x \in C$ is $SN_x(C) = (N_x(C) \setminus \{0\})/\mathbb{R}^{+*}$, where $\mathbb{R}^{+*}$ acts by scalar multiplication. A smooth *submanifold transverse to the ridges* of a smooth manifold A is a subset C of A such that for any point $x \in C$ there exists a smooth open embedding ϕ from $\mathbb{R}^c \times \mathbb{R}^e \times [0,1[^d$ into A such that $\phi(0) = x$ and the image of ϕ intersects C exactly along $\phi(0 \times \mathbb{R}^e \times [0,1[^d)$. Here c is the *codimension* of C, d and e are integers, which depend on x.

For us, *blowing up* such a submanifold C in A replaces C with its unit normal bundle in order to produce the smooth manifold $B\ell(A,C)$ (with possible ridges) so that a chart $\phi\colon \mathbb{R}^c \times \mathbb{R}^e \times [0,1[^d \hookrightarrow A$ as above induces a chart $\overline{\phi}\colon ([0,\infty[\times S^{c-1}) \times \mathbb{R}^e \times [0,1[^d \hookrightarrow B\ell(A,C)$. (The origin 0 of $\mathbb{R}^c$ was replaced with the sphere $\{0\} \times S^{c-1}$ of directions around it.)

Unlike blow-ups in algebraic geometry, this differential geometric blow-up creates boundaries. More precisely, we have the following proposition.

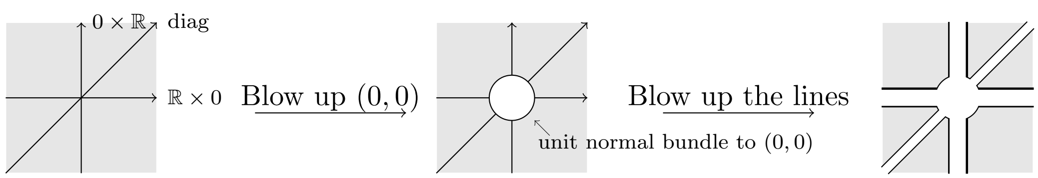
Proposition 1.5 *Under the assumptions of the definition above, we have the following properties.*

- *$B\ell(A,C)$ is diffeomorphic to the complement of an open tubular neighborhood of C (thought of as infinitely small).*
- *There is a canonical projection $p_b\colon B\ell(A,C) \to A$, which restricts to a diffeomorphism from the preimage of $A \setminus C$ to $A \setminus C$.*
- *If A is compact, $B\ell(A,C)$ is a compactification of $A \setminus C$.*
- *If A is closed, then $B\ell(A,C)$ is a compact manifold whose boundary is the unit normal bundle of C in A, and whose interior $B\ell(A,C) \setminus \partial B\ell(A,C)$ is $A \setminus C$.*

Examples 1.6 Local models are given by the following elementary blow-ups $B\ell(\mathbb{R}^c, 0) \cong [0,\infty[\times S^{c-1}$, and $B\ell(\mathbb{R}^c \times A, 0 \times A) \cong [0,\infty[\times S^{c-1} \times A$.

See S^3 as $\mathbb{R}^3 \cup \{\infty\}$ or as two copies of $\mathbb{R}^3$ identified along $\mathbb{R}^3 \setminus \{0\}$ by the (exceptionally orientation-reversing) diffeomorphism $x \mapsto x/\parallel x \parallel^2$. The blow-up $B\ell(S^3,\infty)$ is diffeomorphic to the compact unit ball of $\mathbb{R}^3$. As as set, it reads $B\ell(S^3,\infty) = \mathbb{R}^3 \cup S^2_\infty$ where $(-S^2_\infty)$ denotes the unit normal bundle to ∞ in S^3 and $\partial B\ell(S^3,\infty) = S^2_\infty$. There is a canonical orientation-preserving diffeomorphism $p_\infty\colon S^2_\infty \to S^2$, such that $x \in S^2_\infty$ is the limit of a sequence of points of $\mathbb{R}^3$ approaching ∞ along a line directed by $p_\infty(x) \in S^2$.

In the following figure, we see the result of first blowing up $(0,0)$ in $\mathbb{R}^2$, and next blowing up the closures in $B\ell(\mathbb{R}^2,(0,0))$ of $\{0\} \times \mathbb{R}^*$, $\mathbb{R}^* \times \{0\}$ and the diagonal of $(\mathbb{R}^*)^2$.



Let R be a $\mathbb{Q}$–S^3 equipped with a point $\infty \in R$. Identify a neighborhood of ∞ in R with a neighborhood of ∞ in S^3. Let $\check{R} = R \setminus \{\infty\}$. Let $\check{C}_2(R) = \check{R}^2 \setminus \mathrm{diag}\left(\check{R}^2\right)$. Define the *configuration space* $C_2(R)$ as the compact 6–manifold with boundary and ridges obtained from R^2 by first blowing up (∞,∞) in R^2, and, by next blowing up the closures of $\{\infty\}\times\check{R}$, $\check{R}\times\{\infty\}$ and the diagonal of $\check{R}^2$ in $B\ell(R^2,(\infty,\infty))$.

In particular, $\partial C_2(R)$ contains the unit normal bundle $(\frac{T\check{R}^2}{\mathrm{diag}\left(T\check{R}^2\right)} \setminus \{0\})/\mathbb{R}^{+*}$ to the diagonal of $\check{R}^2$. This bundle is canonically isomorphic to the unit tangent bundle $U\check{R}$ to $\check{R}$ via the map $([(x,y)] \mapsto [y-x])$).

$$\partial C_2(R) = p_b^{-1}(\infty,\infty) \cup (S^2_\infty \times \check{R}) \cup (\check{R} \times S^2_\infty) \cup U\check{R}$$

and

$$\check{C}_2(R) = C_2(R) \setminus \partial C_2(R) = \check{R}^2 \setminus \mathrm{diag}\left(\check{R}^2\right).$$

The following proposition is [Les20, Lemma 3.5].

Proposition 1.7 *The S^2–valued map $p_{S^2}\colon p_{S^2}\colon (x,y) \mapsto \frac{1}{\|y-x\|}(y-x)$ smoothly extends from $\check{C}_2(\mathbb{R}^3)$ to $C_2(S^3)$, and its extension p_{S^2} satisfies:*

$$p_{S^2} = \begin{cases} -p_\infty \circ p_1 & \text{on } S^2_\infty \times \mathbb{R}^3 \\ p_\infty \circ p_2 & \text{on } \mathbb{R}^3 \times S^2_\infty \\ p_2 & \text{on } U\mathbb{R}^3 = \mathbb{R}^3 \times S^2 \end{cases}$$

where p_1 and p_2 denote the projections on the first and second factor with respect to the above expressions.

Also note the following lemma.

Lemma 1.8 *$C_2(S^3)$ is homotopy equivalent to S^2.*

PROOF: $C_2(S^3)$ is homotopy equivalent to its interior $((\mathbb{R}^3)^2 \setminus \text{diag})$, which is homeomorphic to $\mathbb{R}^3 \times]0,\infty[\times S^2$ via the map

$$(x,y) \mapsto (x, \| y-x \|, p_{S^2}(x,y)).$$

□

See $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$, where $\mathbb{C}$ is thought of as horizontal. Let $\mathcal{C}_0 = D^2 \times [0,1]$ be the *standard cyclinder* of $\mathbb{R}^3$, where D^2 is the unit disk of $\mathbb{C}$. Let $\mathcal{C}_0^c$ (resp. $\check{\mathcal{C}}_0^c$) denote the closure of the complement of $\mathcal{C}_0$ in S^3 (resp. in $\mathbb{R}^3$). Here, a *rational homology cylinder* (or *$\mathbb{Q}$–cylinder*) is a compact oriented 3-manifold whose boundary neighborhood is identified with a boundary neighborhood $N(\partial\mathcal{C}_0)$ of $\mathcal{C}_0$, and that has the same rational homology as a point. Any $\mathbb{Q}$–sphere R (may and) will be seen as the union of $\mathcal{C}_0^c$ and of a rational homology cylinder $\mathcal{C}$ glued along $\partial\mathcal{C}_0$. It suffices to choose a point ∞ and a diffeomorphism that identifies a neighborhood of this point in R with $\mathcal{C}_0^c$ to obtain such a decomposition.

Definition 1.9 Let τ_s denote the standard parallelization of $\mathbb{R}^3$. Say that a parallelization

$$\tau\colon \check{R} \times \mathbb{R}^3 \to T\check{R}$$

of $\check{R}$ that coincides with τ_s outside $\mathcal{C}_0$ is *asymptotically standard.* According to [Les20, Proposition 5.5], asymptotically standard parallelizations exist for any R. Such a parallelization identifies $U\check{R}$ with $\check{R} \times S^2$.

An *asymptotic homology* $\mathbb{R}^3$ is a pair $(\check{R},\tau)$ where $\check{R}$ is a punctured rational homology sphere with a decomposition $\check{R} = \mathcal{C} \cup_{\partial\mathcal{C}_0} \check{\mathcal{C}}_0^c$ as above equipped with an asymptotically standard parallelization τ.

Below, we fix such an asymptotic homology $\mathbb{R}^3$ with its decomposition $(\check{R} = \mathcal{C} \cup_{\partial\mathcal{C}_0} \check{\mathcal{C}}_0^c, \tau)$.

Lemma 1.10 *The parallelization τ of $\check{R}$ induces the continuous map $p_\tau\colon \partial C_2(R) \to S^2$ such that*

$$p_\tau = \begin{cases} -p_\infty \circ p_1 & \text{on } S^2_\infty \times \check{R} \\ p_\infty \circ p_2 & \text{on } \check{R} \times S^2_\infty \\ p_2 & \text{on } U\check{R} \stackrel{\tau}{=} \check{R} \times S^2 \\ p_{S^2} & \text{on } p_b^{-1}(\infty,\infty) \end{cases}$$

where p_1 and p_2 denote the projections on the first and second factor with respect to the above expressions.

Proof: This is a corollary of Proposition 1.7. □

Also note the following lemmas.

Lemma 1.11 *$H_*(C_2(R);\mathbb{Q}) \cong H_*(S^2;\mathbb{Q})$ and $H_2(C_2(R);\mathbb{Q})$ is generated by the class $[S]$ of a fiber $U_x\check{R}$ of the bundle $U\check{R}$, oriented as the boundary of a ball of $T_x\check{R}$.*

Proof: The space $C_2(R)$ is homotopy equivalent to its interior $((\check{R})^2 \setminus \text{diag})$, where $\check{R}$ has the rational homology of a point. The rational homology of $((\check{R})^2 \setminus \text{diag})$ can be computed like the rational homology of $((\mathbb{R}^3)^2 \setminus \text{diag})$, which is isomorphic to the rational homology of S^2 thanks to Lemma 1.8. □

Definition 1.12 A *volume one form of* S^2 is a 2-form ω_S of S^2 such that $\int_{S^2}\omega_S = 1$. (See [Les20, Appendix B] for a short survey of differential forms and de Rham cohomology.) Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Recall the map $p_\tau\colon \partial C_2(R) \to S^2$ of Lemma 1.10. A *propagating form* of $(C_2(R),\tau)$ is a closed 2-form ω on $C_2(R)$ whose restriction to $\partial C_2(R)$ reads $p_\tau^*(\omega_S)$ for some volume one form ω_S of S^2. A *propagating chain* of $C_2(R)$ is a rational 4–chain P of $C_2(R)$ such that $\partial P \subset \partial C_2(R)$ and $\partial P \cap \left(\partial C_2(R) \setminus U\check{R}\right) = p^{-1}_{\tau|\partial C_2(R)\setminus U\check{R}}(a)$ for some $a \in S^2$. (This definition does not depend on τ.) A *propagating chain* of $(C_2(R),\tau)$ is a propagating chain of $C_2(R)$ such that $\partial P = p_\tau^{-1}(a)$ for some $a \in S^2$. Propagating chains and propagating forms are simply called *propagators* when their nature is clear from the context.

Example 1.13 Recall the map $p_{S^2}\colon C_2(S^3) \to S^2$ of Proposition 1.7. As already announced, for any $a \in S^2$, $p_{S^2}^{-1}(a)$ is a propagating chain of $(C_2(S^3),\tau_s)$, and for any 2-form ω_S of S^2 such that $\int_{S^2}\omega_S = 1$, $p_{S^2}^*(\omega_S)$ is a propagating form of $(C_2(S^3),\tau_s)$.

For our general $\mathbb{Q}$–sphere R, propagating chains exist because the 3-cycle $p_\tau^{-1}(a)$ of $\partial C_2(R)$ bounds in $C_2(R)$ since $H_3(C_2(R);\mathbb{Q}) = 0$, according to Lemma 1.11. Dually, propagating forms exist because the restriction induces a surjective map $H^2(C_2(R);\mathbb{R}) \to H^2(\partial C_2(R);\mathbb{R})$ since $H^3(C_2(R),\partial C_2(R);\mathbb{R}) = 0$.

When R is a $\mathbb{Z}$-sphere, there exist propagating chains that are smooth 4-manifolds properly embedded in $C_2(R)$. See [Les20, Theorem 11.9]. Explicit propagating chains associated with Heegaard splittings, which were constructed with Greg Kuperberg, are described in Section 1.5 below. They read as integral chains multiplied by $\frac{1}{|H_1(R;\mathbb{Z})|}$, where $|H_1(R;\mathbb{Z})|$ is the cardinality of $H_1(R;\mathbb{Z})$.

Lemma 1.14 *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let C be a two-cycle of $C_2(R)$. For any propagating chain P of $C_2(R)$ transverse to C and for any propagating form ω of $(C_2(R), \tau)$,*

$$[C] = \int_C \omega[S] = \langle C, P \rangle_{C_2(R)}[S]$$

in $H_2(C_2(R); \mathbb{Q}) = \mathbb{Q}[S]$.

PROOF: Fix a propagating chain P, the algebraic intersection $\langle C, P \rangle_{C_2(R)}$ only depends on the homology class $[C]$ of C in $C_2(R)$. Similarly, since ω is closed, $\int_C \omega$ only depends on $[C]$. (Indeed, if C and C' cobound a chain D transverse to P, $C \cap P$ and $C' \cap P$ cobound $\pm(D \cap P)$, and $\int_{\partial D = C' - C} \omega = \int_D d\omega$ according to the Stokes theorem.) Furthermore, the dependence on $[C]$ is linear. Therefore it suffices to check the lemma for a chain that represents the canonical generator $[S]$ of $H_2(C_2(R); \mathbb{Q})$. Any fiber of $U\check{R}$ is such a chain. $\square$

A *meridian* of a knot K is the (oriented) boundary of a disk that intersects K once with a positive sign, as in Figure 2.

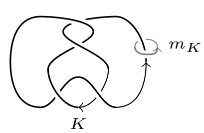


Figure 2: A meridian m_K of a knot K

Lemma 1.15 *Let $J \sqcup K$ be a two-component link of $\check{R}$. The torus $J \times K = (J \times K)(S^1 \times S^1)$ is homologous to $lk(J, K)[S]$ in $H_2(C_2(R); \mathbb{Q})$. For any propagating chain P of $C_2(R)$ transverse to $J \times K$ and for any propagating form ω of $(C_2(R), \tau)$,*

$$lk(J, K) = \int_{J \times K} \omega = \langle J \times K, P \rangle_{C_2(R)}.$$

If $\check{R} = \mathbb{R}^3$, the linking number $lk(J, K)$ of Definition 1.3 is the degree $lk_G(J, K)$ of the Gauss map p_{JK}.

PROOF: When $\check{R} = \mathbb{R}^3$,

$$lk_G(J, K) = \deg_a(p_{JK}) = \langle J \times K, p_{S^2}^{-1}(a) \rangle_{C_2(S^3)}$$

so that $J \times K$ is homologous to $lk_G(J, K)[S]$ in $H_2(C_2(S^3); \mathbb{Q})$ according to Lemma 1.14, with the propagator $p_{S^2}^{-1}(a)$ of Example 1.13. For an arbitrary $\check{R}$, define $lk_G(J, K)$ so that $J \times K$ is homologous to $lk_G(J, K)[S]$ in $H_2(C_2(R); \mathbb{Q})$. Recall from Definition 1.3 that $lk(J, K)$ is the algebraic intersection $\langle J, \Sigma_K \rangle_R$ of J and a rational chain Σ_K bounded by K. Lemma 1.14

reduces the proof of Lemma 1.15 to the proof that $lk(J,K)$ and $lk_G(J,K)$ coincide for any two-component link $J \sqcup K$ of $\check{R}$. Note that the definitions of $lk(J,K)$ and $lk_G(J,K)$ make sense when J and K are disjoint links. If J has several components J_i, for $i=1,\dots,n$, then $lk_G(\sqcup_{i=1}^n J_i, K) = \sum_{i=1}^n lk_G(J_i,K)$ and $lk(\sqcup_{i=1}^n J_i, K) = \sum_{i=1}^n lk(J_i,K)$. There is no loss in assuming that J is a knot for the proof, and we do. The chain Σ_K provides a rational cobordism C in $\check{R} \setminus J$ between K and a combination of meridians of J, which is homologous to $lk(J,K)[m_J]$. The product rational cobordism $J \times C$ in $\check{R}^2 \setminus \operatorname{diag}\left(\check{R}^2\right)$ allows us to see that $[J \times K] = lk(J,K)[J \times m_J]$ in $H_2(\check{R}^2 \setminus \operatorname{diag}\left(\check{R}^2\right); \mathbb{Q})$. Similarly, a chain Σ_J bounded by J provides a rational cobordism between J and a meridian m_{m_J} of m_J so that $[J \times m_J] = [m_{m_J} \times m_J]$ in $H_2(\check{R}^2 \setminus \operatorname{diag}\left(\check{R}^2\right); \mathbb{Q})$, and $lk_G(J,K) = lk(J,K) lk_G(m_{m_J}, m_J)$. Thus we are left with the proof that $lk_G(m_{m_J}, m_J) = 1$ for a positive Hopf link (m_{m_J}, m_J) in a standard ball embedded in $\check{R}$. Now, there is no loss in assuming that our link is a Hopf link in $\mathbb{R}^3$ so that the equality follows from the equality for the positive Hopf link in $\mathbb{R}^3$. □

Lemma 1.15 shows in what sense *propagators represent the linking number*. We are going to use these propagators to define invariants of $\mathbb{Q}$–spheres, below.

1.4 On the Theta invariant

More on algebraic intersections The intersection of two transverse submanifolds A and B in a manifold M is a manifold, which is oriented so that the normal bundle to $A \cap B$ is $(N(A) \oplus N(B))$, fiberwise. In order to give a meaning to the sum $(N_x(A) \oplus N_x(B))$ at $x \in A \cap B$, pick a Riemannian metric on M, which canonically identifies $N_x(A)$ with $T_x(A)^\perp$, $N_x(B)$ with $T_x(B)^\perp$ and $N_x(A \cap B)$ with $T_x(A \cap B)^\perp = T_x(A)^\perp \oplus T_x(B)^\perp$. Since the space of Riemannian metrics on M is convex, and therefore connected, the induced orientation of $T_x(A \cap B)$ does not depend on the choice of Riemannian metric.

Let A, B, C be three pairwise transverse submanifolds in a manifold M such that $A \cap B$ is transverse to C. The oriented intersection $(A \cap B) \cap C$ is a well-defined manifold. Our assumptions imply that at any $x \in A \cap B \cap C$, the sum $(T_xA)^\perp + (T_xB)^\perp + (T_xC)^\perp$ is a direct sum $(T_xA)^\perp \oplus (T_xB)^\perp \oplus (T_xC)^\perp$ for any Riemannian metric on M so that A is also transverse to $B \cap C$, and $(A \cap B) \cap C = A \cap (B \cap C)$. Thus, the intersection of transverse, oriented submanifolds is a well-defined associative operation, where *transverse submanifolds* are manifolds such that the elementary pairwise intermediate possible intersections are well-defined as above. This intersection is also commutative when the codimensions of the submanifolds are even.

The *algebraic intersection* of several transverse compact submanifolds $A_1, \dots, A_k$ of M whose codimension sum is the dimension of M is $\langle A_1, \dots, A_k \rangle_M = \langle \cap_{i=1}^{k-1} A_i, A_k \rangle_M$. If M is a connected manifold, which contains a point x, the class of a 0-cycle in $H_0(M;\mathbb{Q}) = \mathbb{Q}[x] = \mathbb{Q}$ is a well-defined number, and $\langle A_1, \dots, A_k \rangle_M$ can equivalently be defined as the homology class of the (oriented) intersection $\cap_{i=1}^k A_i$. This algebraic intersection extends to rational chains, multilinearly.

Theorem 1.16 *Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let P_a, P_b and P_c be three transverse propagating chains of $(C_2(R),\tau)$ with respective boundaries $p_\tau^{-1}(a)$, $p_\tau^{-1}(b)$ and $p_\tau^{-1}(c)$ for three distinct points a, b and c of S^2, then*

$$\Theta(R,\tau) = \langle P_a, P_b, P_c\rangle_{C_2(R)}$$

does not depend on the chosen propagators P_a, P_b and P_c. It is a topological invariant of (R,τ).

PROOF: Since $H_4(C_2(R);\mathbb{Q}) = 0$, if the propagator P_a is changed to a propagator P'_a with the same boundary, $(P'_a - P_a)$ bounds a 5-dimensional rational chain W transverse to $P_b \cap P_c$. The 1-dimensional chain $W \cap P_b \cap P_c$ does not meet $\partial C_2(R)$ since $P_b \cap P_c$ does not meet $\partial C_2(R)$. Therefore, up to a well-determined sign, the boundary of $W \cap P_b \cap P_c$ is $P'_a \cap P_b \cap P_c - P_a \cap P_b \cap P_c$. This shows that $\langle P_a, P_b, P_c\rangle_{C_2(R)}$ is independent of P_a when a is fixed. Similarly, it is independent of P_b and P_c when b and c are fixed. Thus, $\langle P_a, P_b, P_c\rangle_{C_2(R)}$ is a rational function on the connected set of triples (a,b,c) of distinct point of S^2. It is easy to see that this function is continuous. Thus, it is constant. □

Lemma 1.17 *Let ω_a and ω'_a be two propagating forms of $(C_2(R),\tau)$, which restrict to $\partial C_2(R)$ as $p_\tau^*(\omega_A)$ and $p_\tau^*(\omega'_A)$, respectively, for two volume one forms ω_A and ω'_A of S^2. There exists a one-form η_A on S^2 such that $\omega'_A = \omega_A + d\eta_A$. For any such η_A, there exists a one-form η on $C_2(R)$ such that $\omega'_a - \omega_a = d\eta$, and the restriction of η to $\partial C_2(R)$ is $p_\tau^*(\eta_A)$.*

PROOF OF THE LEMMA: Since ω_a and ω'_a are cohomologous, there exists a one-form η on $C_2(R)$ such that $\omega'_a = \omega_a + d\eta$. Similarly, since $\int_{S^2}\omega'_A = \int_{S^2}\omega_A$, there exists a one-form η_A on S^2 such that $\omega'_A = \omega_A + d\eta_A$. On $\partial C_2(R)$, $d(\eta - p_\tau^*(\eta_A)) = 0$. Thanks to the exact sequence with real coefficients

$$0 = H^1(C_2(R)) \longrightarrow H^1(\partial C_2(R)) \longrightarrow H^2(C_2(R),\partial C_2(R)) \cong H_4(C_2(R)) = 0,$$

$H^1(\partial C_2(R);\mathbb{R}) = 0$. Therefore, there exists a function f from $\partial C_2(R)$ to $\mathbb{R}$ such that

$$df = \eta - p_\tau^*(\eta_A)$$

on $\partial C_2(R)$. Extend f to a C^∞ map on $C_2(R)$ and change η to $(\eta - df)$. □

Theorem 1.18 *Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. For any three propagating forms ω_a, ω_b and ω_c of $(C_2(R),\tau)$,*

$$\Theta(R,\tau) = \int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c.$$

PROOF: Let us first prove that $\int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c$ is independent of the propagating forms ω_a, ω_b and ω_c. Using Lemma 1.17 and its notations

$$\begin{aligned}\int_{C_2(R)} \omega'_a \wedge \omega_b \wedge \omega_c - \int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c &= \int_{C_2(R)} d(\eta \wedge \omega_b \wedge \omega_c) \\ &= \int_{\partial C_2(R)} \eta \wedge \omega_b \wedge \omega_c \\ &= \int_{\partial C_2(R)} p_\tau^*(\eta_A \wedge \omega_B \wedge \omega_C) = 0\end{aligned}$$

since any 5-form on S^2 vanishes. Thus, $\int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c$ is independent of the propagating forms ω_a, ω_b and ω_c. Now, we can choose the propagating forms ω_a, ω_b and ω_c supported in very small neighborhoods of P_a, P_b and P_c and Poincaré dual to P_a, P_b and P_c, respectively, so that the intersection of the three supports is a very small neighborhood of $P_a \cap P_b \cap P_c$, where it can easily be seen that $\int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c = \langle P_a, P_b, P_c \rangle_{C_2(R)}$. See [Les20, Section 11.4, Section B.2 and Lemma B.4] in particular, for more details. □

In particular, $\Theta(R,\tau)$ reads $\int_{C_2(R)} \omega^3$ for any propagating form ω of $(C_2(R), \tau)$. Since such a propagating form represents the linking number, $\Theta(R,\tau)$ can be thought of as the *cube of the linking number with respect to τ*. When τ varies continuously, $\Theta(R,\tau)$ varies continuously in $\mathbb{Q}$ so that $\Theta(R,\tau)$ is an invariant of the homotopy class of τ.

Example 1.19 Using (disjoint !) propagators $p_{S^2}^{-1}(a)$, $p_{S^2}^{-1}(b)$, $p_{S^2}^{-1}(c)$ associated to three distinct points a, b and c of $\mathbb{R}^3$, as in Example 1.13, it is clear that

$$\Theta(S^3, \tau_s) = \langle p_{S^2}^{-1}(a), p_{S^2}^{-1}(b), p_{S^2}^{-1}(c) \rangle_{C_2(S^3)} = 0.$$

Parallelizations of 3-manifolds and Pontrjagin classes

Definition 1.20 Let $SO(3)$ be the group of orientation-preserving linear isometries of $\mathbb{R}^3$. In this paragraph, see S^3 as $B^3/\partial B^3$ where B^3 is the standard unit ball of $\mathbb{R}^3$ seen as $([0,1] \times S^2)/(0 \sim \{0\} \times S^2)$. Let $\chi_\pi \colon [0,1] \to [0, 2\pi]$ be an increasing smooth bijection whose derivatives vanish at 0 and 1 such that $\chi_\pi(1-\theta) = 2\pi - \chi_\pi(\theta)$ for any $\theta \in [0,1]$. Define the map $\rho \colon B^3 \to SO(3)$ that maps $(\theta \in [0,1], v \in S^2)$ to the rotation $\rho(\chi_\pi(\theta); v)$ with axis directed by v and with angle $\chi_\pi(\theta)$.

This map[6] induces the double covering $\tilde{\rho} \colon S^3 \to SO(3)$, which identifies $SO(3)$ with the real projective space $\mathbb{R}P^3$, and which orients $SO(3)$.

For any map g from $\check{R}$ to $SO(3)$ that sends $\check{\mathcal{C}}_0^c$ to the unit $1_{SO(3)}$ of $SO(3)$, define

$$\begin{array}{rccc}\psi_{\mathbb{R}}(g): & \check{R} \times \mathbb{R}^3 & \longrightarrow & \check{R} \times \mathbb{R}^3 \\ & (x,y) & \mapsto & (x, g(x)(y)).\end{array}$$

[6]This double covering map allows one to deduce the first three homotopy groups of $SO(3)$ from the ones of S^3. The first three homotopy groups of $SO(3)$ are $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$, $\pi_2(SO(3)) = 0$ and $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$. For $v \in S^2$, $\pi_1(SO(3))$ is generated by the class of the loop that maps $\exp(i\theta) \in S^1$ to the rotation $\rho(\theta; v)$. See [Les20, Section A.2 and Theorem A.13, in particular].

Since $GL^+(\mathbb{R}^3)$ deformation retracts onto $SO(3)$, any asymptotically standard parallelization of $\check{R}$ is homotopic to $\tau \circ \psi_{\mathbb{R}}(g)$ for some g as above.

The following classical theorem is proved in [Les20, Chapter 5]. See Proposition 5.21 in particular.

Theorem 1.21 *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. There exists a canonical map p_1 from the set of homotopy classes of asymptotically standard parallelizations of $\check{R}$ to $\mathbb{Z}$ such that $p_1(\tau_s) = 0$, and, for any map g from R to $SO(3)$ that sends $\mathcal{C}_0^c$ to the unit $1_{SO(3)}$ of $SO(3)$*

$$p_1(\tau \circ \psi_{\mathbb{R}}(g_{|\check{R}})) - p_1(\tau) = 2deg(g).$$

The definition of the map p_1 is given in [Les20, Section 5.5], it involves relative Pontrjagin classes. See [Les20, Proposition 5.10]. It is similar to the map h studied by Hirzebruch in [Hir73, §3.1], and by Kirby and Melvin in [KM99] under the name of *Hirzebruch defect.*

The following proposition is proved in [Les20, Section 4.3]. See Proposition 4.8.

Proposition 1.22 *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. For any map g from R to $SO(3)$ that sends $\mathcal{C}_0^c$ to $1_{SO(3)}$,*

$$\Theta(R, \tau \circ \psi_{\mathbb{R}}(g_{|\check{R}})) - \Theta(R, \tau) = \frac{1}{2} deg(g).$$

Theorem 1.21 allows us to derive the following corollary from Proposition 1.22.

Corollary 1.23 $\Theta(R) = \Theta(R, \tau) - \frac{1}{4}p_1(\tau)$ *is an invariant of $\mathbb{Q}$-spheres.*

□

The invariant Θ coincides with $6\lambda_{CW}$ where λ_{CW} denotes the Casson-Walker invariant. This Walker invariant generalizes the Casson invariant of $\mathbb{Z}$-spheres, which counts the conjugacy classes of irreducible representations of their fundamental groups using Heegaard splittings. See [AM90, GM92, Mar88]. It is normalized like in [AM90, GM92, Mar88] for integer homology 3-spheres, and like $\frac{1}{2}\lambda_W$ for rational homology 3-spheres where λ_W is the Walker normalisation in [Wal92]. The equality ($\Theta = 6\lambda_{CW}$) was proved by Kuperberg and Thurston in [KT99] for $\mathbb{Z}$–spheres, and it was generalized to $\mathbb{Q}$-spheres in [Les04, Section 6]. See [Les04, Theorem 2.6] or [Les20, Theorem 17.25].

The main part of the proof consists in comparing second derivatives or (variations of variations) of Θ and λ_{CW} under the following *Lagrangian-preserving surgeries.*

Lagrangian-preserving surgeries

Definition 1.24 An *integer (resp. rational) homology handlebody* of genus g is a compact oriented 3-manifold A that has the same integral (resp. rational) homology as the usual solid handlebody H_g of Figure 3.

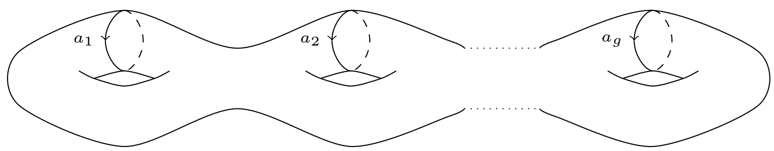


Figure 3: The standard handlebody H_g

Exercise 1.25 Show that if A is a rational homology handlebody of genus g, then ∂A is a genus g surface.

The *Lagrangian* $\mathcal{L}_A$ of a compact 3-manifold A is the kernel of the map induced by the inclusion from $H_1(\partial A;\mathbb{Q})$ to $H_1(\partial A;\mathbb{Q})$.

In Figure 3, the Lagrangian of H_g is freely generated by the classes of the curves a_i.

Definition 1.26 An *integral (resp. rational) Lagrangian-Preserving (or LP) surgery* (A'/A) is the replacement of an integral (resp. rational) homology handlebody A embedded in the interior of a 3-manifold M with another such A' whose boundary is identified with ∂A by an orientation-preserving diffeomorphism that sends $\mathcal{L}_A$ to $\mathcal{L}_{A'}$. The manifold $M(A'/A)$ obtained by such an LP-surgery reads[7]

$$M(A'/A) = (M \setminus \mathrm{Int}(A)) \cup_{\partial A} A'.$$

Lemma 1.27 *If (A'/A) is an integral (resp. rational) LP-surgery in a 3-manifold M, then the homology of $M(A'/A)$ with $\mathbb{Z}$-coefficients (resp. with $\mathbb{Q}$-coefficients) is canonically isomorphic to $H_*(M;\mathbb{Z})$ (resp. to $H_*(M;\mathbb{Q})$). If M is a $\mathbb{Q}$-sphere, if (A'/A) is a rational LP-surgery, and if (J,K) is a two-component link of $M \setminus A$, then the linking number of J and K in M and the linking number of J and K in $M(A'/A)$ coincide.*

Proof: Exercise. □

In [Les04], I computed

$$\Theta\left(R\left(A'/A, B'/B\right)\right) - \Theta\left(R\left(A'/A\right)\right) - \Theta\left(R\left(B'/B\right)\right) + \Theta(R)$$

and proved that it coincides with

$$6\lambda_{CW}\left(R\left(A'/A, B'/B\right)\right) - 6\lambda_{CW}\left(R\left(A'/A\right)\right) - 6\lambda_{CW}\left(R\left(B'/B\right)\right) + 6\lambda_{CW}(R)$$

for any two rational LP-surgeries (A'/A) and (B'/B) in a $\mathbb{Q}$–sphere R such that A and B are disjoint rational homology handlebodies in $\check{R}$. Together with the property that $\Theta(-R) =$

[7]This description only defines the topological structure of $M(A'/A)$, but we equip $M(A'/A)$ with its unique smooth structure.

$-\Theta(R)$, this implies that $\Theta = 6\lambda_{CW}$. See [Les20, Theorem 17.25]. In order to perform the computation of the above discrete "second derivative"

$$(\Theta(R(A'/A, B'/B)) - \Theta(R(B'/B))) - (\Theta(R(A'/A)) - \Theta(R))$$

of Θ, I built propagators for the 4 involved $\mathbb{Q}$–spheres, which coincide as much as possible.

1.5 A propagator associated to a Heegaard diagram

In this section, we give an example of a propagating chain associated to a Heegaard diagram or to a self-indexed Morse function of an asymptotic homology $\mathbb{R}^3$. I constructed such a *Morse propagator* with Greg Kuperberg in [Les15a]. Similar propagators associated to more general Morse functions have been constructed by Watanabe in [Wat18], independently.

First note the propagator $p_{S^2}^{-1}(\vec{N})$ associated to the upward vertical vector $\vec{N}$ intersects $(\mathbb{R}^3)^2 \setminus \mathrm{diag}$ as $\{(x, x+t\vec{N}) \mid x \in \mathbb{R}^3, t \in]0,+\infty]\}$. The explicit propagator that we are about to construct for an asymptotic homology $\check{R}$ is built from the closure P_ϕ in $C_2(R)$ of $\{(x, \phi_t(x)) \mid x \in \check{R}, t \in]0,+\infty]\}$, where (ϕ_t) is the flow associated to a Morse function without minima and maxima of $\check{R}$ and to a metric $\mathfrak{g}$ on $\check{R}$.

Start with $\mathbb{R}^3$ equipped with its standard height function f_0 and replace the cube $[-\frac{1}{2},\frac{1}{2}]^2 \times [0,1]$ with a rational homology cube C_R (which has the rational homology of a point) equipped with a Morse function f, which coincides with f_0 on $\partial\left([-\frac{1}{2},\frac{1}{2}]^2 \times [0,1]\right)$, and which has $2g$ critical points, g points $a_1, \dots, a_g$ of index 1, mapped to $1/3$ by f, and g points $b_1, \dots, b_g$ of index 2, mapped to $2/3$ by f (so that $3f$ is self-indexed). Let $\check{R}$ be the associated open manifold, and let R be its one-point compactification. Equip $\check{R}$ with a Riemannian metric $\mathfrak{g}$ that coincides with the standard one outside $[-\frac{1}{2},\frac{1}{2}]^2 \times [0,1]$.

The preimage H_a of $]-\infty,\frac{1}{2}]$ under f in C_R has the standard representation of the bottom part of Figure 4. Our standard representation of the preimage H_b of $[\frac{1}{2},+\infty[$ under f in C_R is shown in the upper part of Figure 4. These two pieces are equipped with standard Morse functions and metrics, a few corresponding flow lines are drawn in Figure 5. They are glued to each other by an a priori non trivial diffeomorphism of ∂H_a.

The two-dimensional ascending manifold of a_i is oriented arbitrarily, its closure is denoted by $\mathcal{A}_i$. Its intersection with H_a is denoted by $D(\alpha_i)$. The boundary of $D(\alpha_i)$ is denoted by α_i. The descending manifold of a_i is made of two half-lines $\mathcal{L}_+(a_i)$ and $\mathcal{L}_-(a_i)$ starting as vertical lines and ending at a_i. The one with the orientation of the positive normal to $\mathcal{A}_i$ is called $\mathcal{L}_+(a_i)$. Thus $\mathcal{L}(a_i) = \mathcal{L}_+(a_i) \cup (-\mathcal{L}_-(a_i))$ is the descending manifold of a_i.

Symmetrically, the two-dimensional descending manifold of b_j is oriented arbitrarily, its closure is denoted by $\mathcal{B}_j$. The $\mathcal{B}_j$ are assumed to be transverse to the $\mathcal{A}_i$ outside the critical points. The intersection $H_b \cap \mathcal{B}_j$ is denoted by $D(\beta_j)$. The boundary of $D(\beta_j)$ is denoted by β_j. The ascending manifold of b_j is made of two half-lines $\mathcal{L}_+(b_j)$ and $\mathcal{L}_-(b_j)$ starting at b_j and ending as vertical lines. The one with the orientation of the positive normal to $\mathcal{B}_j$ is called $\mathcal{L}_+(b_j)$. Thus $\mathcal{L}(b_j) = \mathcal{L}_+(b_j) - \mathcal{L}_-(b_j)$ is the ascending manifold of b_j. See Figure 5. Let

$$[\mathcal{J}_{ji}]_{(j,i)\in\{1,\dots,g\}^2} = [\langle \alpha_i, \beta_j \rangle_{\partial H_a}]^{-1}$$

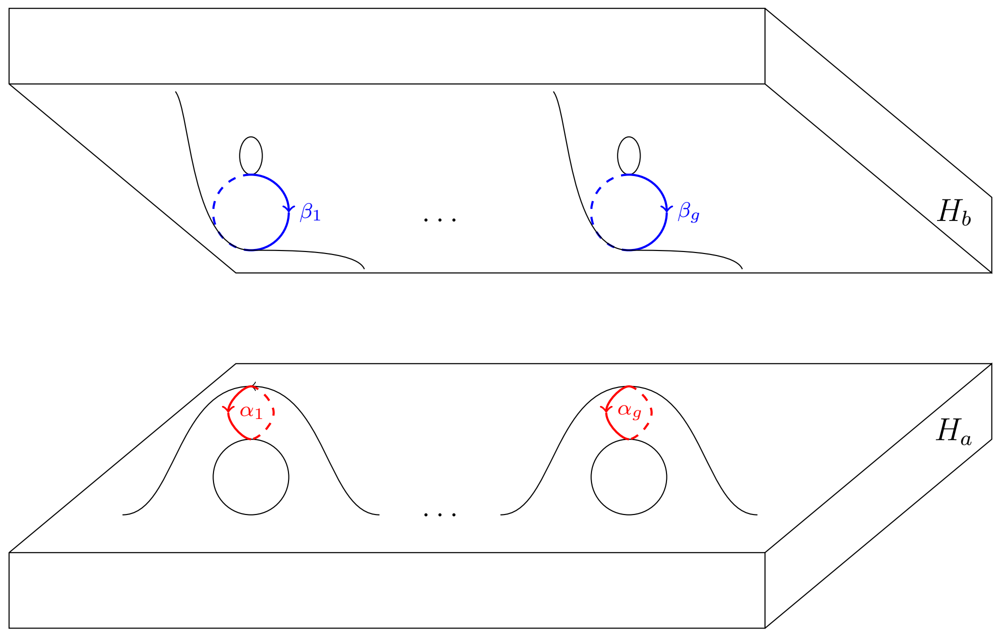


Figure 4: H_a and H_b

be the inverse matrix of the matrix of the algebraic intersection numbers $\langle \alpha_i, \beta_j \rangle_{\partial H_a}$.

Let ϕ be the flow associated to the gradient of f and to $\mathfrak{g}$. Let P_ϕ be the closure in $C_2(R)$ of the image of

$$\begin{array}{lcl} \left(\check{R} \setminus \{a_i, b_i; i \in \{1, \ldots, g\}\}\right) \times]0, +\infty[& \to & C_2(R) \\ (x, t) & \mapsto & (x, \phi_t(x)), \end{array}$$

let $((\mathcal{B}_j \times \mathcal{A}_i) \cap C_2(R))$ denote the closure of $\left((\mathcal{B}_j \times \mathcal{A}_i) \cap (\check{R}^2 \setminus \text{diag})\right)$ in $C_2(R)$, set

$$P_{\mathcal{I}} = \sum_{(i,j) \in \{1,\ldots,g\}^2} \mathcal{J}_{ji} \left((\mathcal{B}_j \times \mathcal{A}_i) \cap C_2(R)\right) \qquad \text{and} \qquad P(f, \mathfrak{g}) = P_\phi + P_{\mathcal{I}}$$

The following proposition is proved in [Les15a]. See Theorem 4.2.

Proposition 1.28 (Kuperberg–Lescop) *The chain $P(f, \mathfrak{g})$ is a propagating chain of $C_2(R)$.*

In particular, $P(f, \mathfrak{g})$ can be used to compute linking numbers as in Lemma 1.15. It suffices to correct the boundary of $P(f, \mathfrak{g})$ near the boundary of $C_2(R)$ to transform $P(f, \mathfrak{g})$ into a propagator of $(C_2(R), \tau)$ as in Definition 1.12.

Define a *combing* of $\check{R}$ as a section of $U\check{R}$ which is constant outside $\mathcal{C}_0$. For such a combing X, a *propagating chain* of $(C_2(R), X)$ is a propagating chain P of $C_2(R)$ such that $P \cap U\check{R} = X(\check{R})$. Define $\tilde{\Theta}(R, X)$ as the algebraic intersection of a propagating chain of $(C_2(R), X)$, a propagating

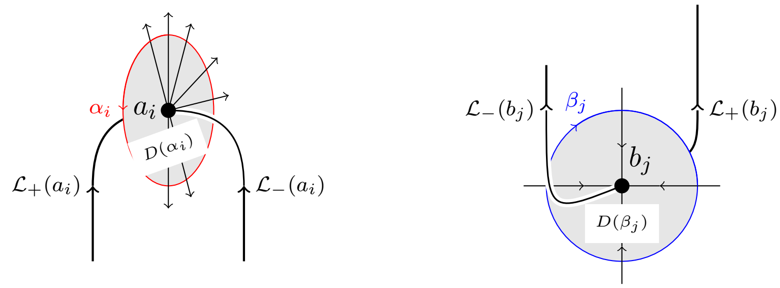


Figure 5: $\mathcal{L}_+(a_i)$, $\mathcal{L}_-(a_i)$, $\mathcal{L}_+(b_j)$, $\mathcal{L}_-(b_j)$

chain of $(C_2(R), -X)$ and any other propagating chain. It is easy to see that $\tilde{\Theta}(R,.)$ is a homotopy invariant of combings (see [Les15a, Theorem 2.1]) and that $\Theta(R,\tau) = \tilde{\Theta}(R,\tau(.,v))$, for any unit vector v of $\mathbb{R}^3$. Further properties of the invariant $\tilde{\Theta}(R,.)$ of combings are studied in [Les15b]. An explicit formula for the invariant $\tilde{\Theta}(R,.)$ from a Heegaard diagram of R was found by the author in [Les15a]. See [Les15a, Theorem 3.8]. It was directly computed using the above definition of $\tilde{\Theta}(R,.)$ together with the above Morse propagators, corrected near the boundary as in [Les15a, Section 5].

2 Configuration space integrals

2.1 Jacobi diagrams and associated configuration space integrals

Definition 2.1 A *uni-trivalent graph* Γ is a 6-tuple

$$(H(\Gamma), E(\Gamma), U(\Gamma), T(\Gamma), p_E, p_V)$$

where $H(\Gamma)$, $E(\Gamma)$, $U(\Gamma)$ and $T(\Gamma)$ are finite sets, which are called the set of half-edges of Γ, the set of edges of Γ, the set of univalent vertices of Γ and the set of trivalent vertices of Γ, respectively, $p_E \colon H(\Gamma) \to E(\Gamma)$ is a two-to-one map (every element of $E(\Gamma)$ has two preimages under p_E) and $p_V \colon H(\Gamma) \to U(\Gamma) \sqcup T(\Gamma)$ is a map such that every element of $U(\Gamma)$ has one preimage under p_V and every element of $T(\Gamma)$ has three preimages under p_V, up to isomorphism. In other words, Γ is a set $H(\Gamma)$ equipped with two partitions, a partition into pairs (induced by p_E), and a partition into singletons and triples (induced by p_V), up to the bijections that preserve the partitions. These bijections are the *automorphisms* of the uni-trivalent graph Γ.

Definition 2.2 Let $\mathcal{L}$ be a non-necessarily oriented one-manifold. A *Jacobi diagram* Γ *with support* $\mathcal{L}$, also called *Jacobi diagram on* $\mathcal{L}$, is a finite uni-trivalent graph Γ equipped with an isotopy class $[i_\Gamma]$ of injections i_Γ from the set $U(\Gamma)$ of univalent vertices of Γ into the interior of $\mathcal{L}$. For such a Γ, a Γ-*compatible injection* is an injection in the class $[i_\Gamma]$.

A Jacobi diagram Γ is represented by a planar immersion of $\Gamma \cup \mathcal{L} = \Gamma \cup_{U(\Gamma)} \mathcal{L}$ where the univalent vertices of $U(\Gamma)$ are located at their images under a Γ-compatible injection i_Γ, the one-manifold $\mathcal{L}$ is represented by dashed lines, whereas the edges of the diagram Γ are represented by plain segments. (The one-manifold $\mathcal{L}$ may be oriented in order to fix the isotopy class $[i_\Gamma]$.)

Figure 6 shows an example of a picture of a Jacobi diagram.

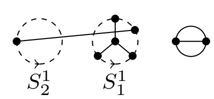


Figure 6: A Jacobi diagram Γ on the disjoint union $\mathcal{L} = S^1_1 \sqcup S^1_2$ of two (oriented) circles

Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $\mathcal{L}$ be a one-manifold and let

$$L : \mathcal{L} \longrightarrow \check{R}$$

denote a C^∞ embedding from $\mathcal{L}$ to $\check{R}$. Let Γ be a Jacobi diagram with support $\mathcal{L}$ as in Definition 2.2. Let $U = U(\Gamma)$ denote the set of univalent vertices of Γ, and let $T = T(\Gamma)$ denote the set of trivalent vertices of Γ. A *configuration* of Γ is an injection

$$c : U \cup T \hookrightarrow \check{R}$$

whose restriction $c_{|U}$ to U may be written as $L \circ j$ for some Γ-compatible injection

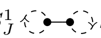
$$j : U \hookrightarrow \mathcal{L}.$$

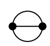
Denote the set of these configurations by $\check{C}(R, L; \Gamma)$ (or $\check{C}(L; \Gamma)$, when R is known or part of the data).

$$\check{C}(R, L; \Gamma) = \left\{ c : U \cup T \hookrightarrow \check{R} \mid \exists j \in [i_\Gamma], c_{|U} = L \circ j \right\}.$$

In $\check{C}(R, L; \Gamma)$, the univalent vertices move along $L(\mathcal{L})$ while the trivalent vertices move in the ambient space $\check{R}$, and $\check{C}(R, L; \Gamma)$ is naturally an open submanifold of $\mathcal{L}^U \times \check{R}^T$. When the ambient asymptotic rational homology $\mathbb{R}^3$ is $\mathbb{R}^3$, we write $\check{C}(L; \Gamma) = \check{C}(S^3, L; \Gamma)$.

Examples 2.3 For a two-component link $J \sqcup K \colon S^1 \sqcup S^1 \to \check{R}$,

$\check{C}\left(R, J \sqcup K; S^1_J\right.$  $\left.S^1_K\right) = J \times K.$

$\check{C}\left(R, \emptyset;\right.$  $\left.\right) = \check{R}^2 \setminus \mathrm{diag}\left(\check{R}^2\right) = \check{C}_2(R).$

Recall that R is seen as the union of $\mathcal{C}_0^c$ and of a rational homology cylinder $\mathcal{C}$ glued along $\partial\mathcal{C}_0$ as before Definition 1.9.

Definition 2.4 A *long tangle representative* in $\check{R}$ is an embedding $L \colon \mathcal{L} \hookrightarrow \check{R}$ of a one-manifold $\mathcal{L}$, as in Figure 7 or Figure 8, such that

- $$L(\mathcal{L}) \cap \check{\mathcal{C}}_0^c = (c^-(B^-) \times]-\infty, 0]) \cup (c^+(B^+) \times [1, \infty[)$$

  for two finite sets B^- and B^+ and two injective maps $c^- \colon B^- \hookrightarrow \mathrm{Int}(D^2)$ $c^+ \colon B^+ \hookrightarrow \mathrm{Int}(D^2)$, which are respectively called the *bottom configuration* and the *top configuration* of L, and
- $L(\mathcal{L}) \cap \mathcal{C}$ is a compact one-manifold whose unoriented boundary is $(c^-(B^-) \times \{0\}) \cup (c^+(B^+) \times \{1\})$.

Figure 9 shows an example of a Jacobi diagram Γ on its source $\mathcal{L}$ together with a configuration of $\check{C}(R, L; \Gamma)$ (where the edges are only drawn to identify the vertices, the configuration is determined by the images of the vertices).

Definition 2.5 An *orientation* of a trivalent vertex of Γ is a cyclic order on the set of the three half-edges that meet at this vertex. An *orientation* of a univalent vertex u of Γ is an orientation of the connected component $\mathcal{L}(u)$ of $i_\Gamma(u)$ in $\mathcal{L}$, for a choice of Γ-compatible i_Γ, associated to u. This orientation is also called (and thought[8] of as) a *local orientation of* $\mathcal{L}$ *at* u.

A *vertex-orientation* of a Jacobi diagram Γ is an *orientation* of every vertex of Γ. A Jacobi diagram is *oriented* if it is equipped with a vertex-orientation[9].

[8] A *local orientation* of $\mathcal{L}$ is simply an orientation of $\mathcal{L}(u)$, but since different vertices are allowed to induce different orientations, we think of these orientations as being *local*, i.e. defined in a neighborhood of $i_\Gamma(u)$ for a choice of Γ-compatible i_Γ.

[9] When $\mathcal{L}$ is oriented, it suffices to specify the orientations of the trivalent vertices since the univalent vertices are oriented by $\mathcal{L}$.

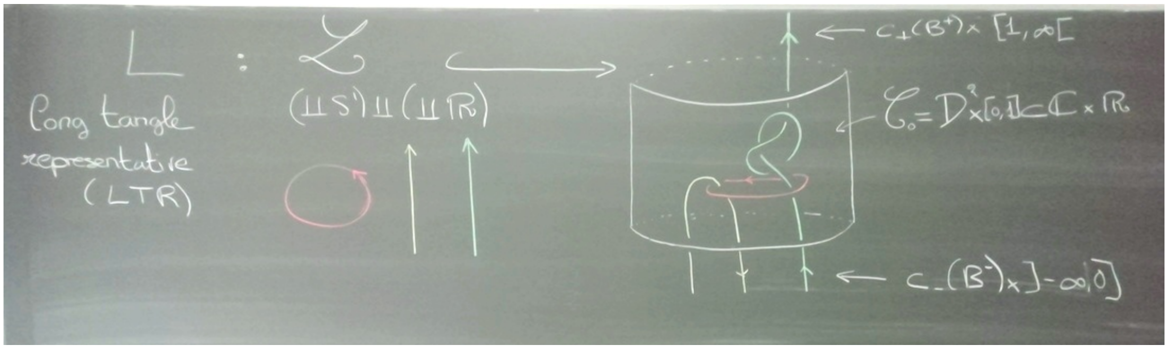


Figure 7: A long tangle representative (LTR) in $\mathbb{R}^3$ (from the lectures)

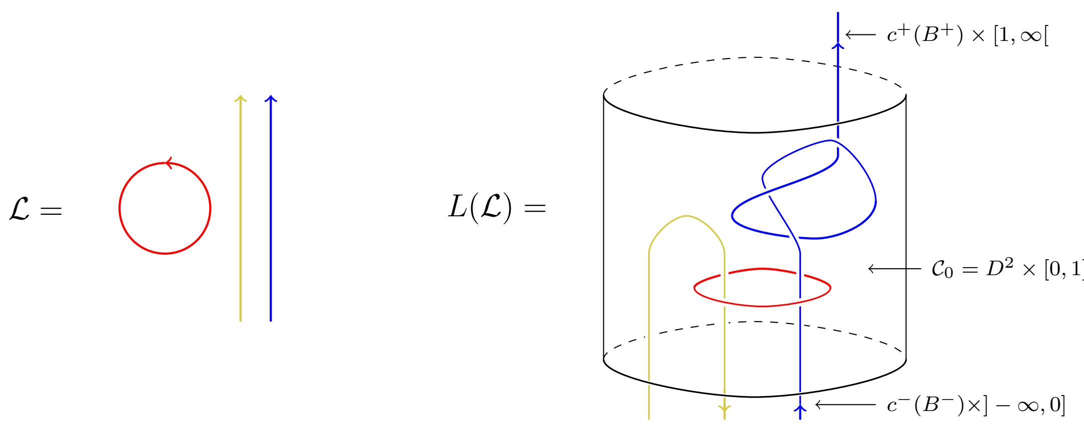


Figure 8: A long tangle representative (LTR) in $\mathbb{R}^3$

In figures, the orientation of a trivalent vertex is represented by the counterclockwise order of the three half-edges that meet at it. The orientation of a univalent vertex u of a Jacobi diagram on a (non-oriented) one-manifold $\mathcal{L}$ corresponds to the counterclockwise cyclic order of the three half-edges that meet at u in a planar immersion of $\Gamma \cup_{U(\Gamma)} \mathcal{L}$ where the half-edge of u in Γ is attached to the left-hand side of $\mathcal{L}$, with respect to the local orientation of $\mathcal{L}$ at u, as in the following pictures.

An *orientation* of a set X of cardinality at least 2 is a total order of the elements of X up to an even permutation.

Cut each edge of Γ into two half-edges. When an edge is oriented, define its *first* half-edge and its *second* one, so that following the orientation of the edge, the first half-edge is met first. Recall that $H(\Gamma)$ denotes the set of half-edges of Γ. When the edges of Γ are oriented, the orientations of the edges of Γ induce the following orientation of the set $H(\Gamma)$ of half-edges of

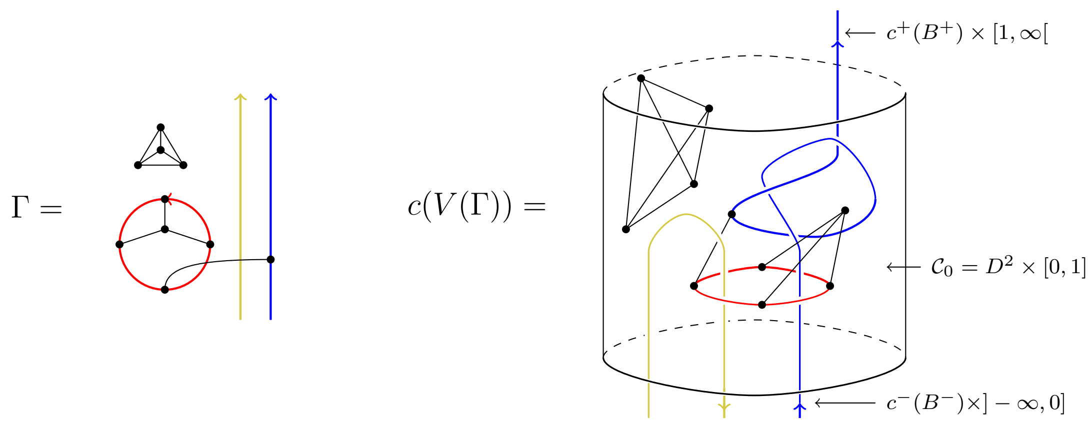


Figure 9: A (black) Jacobi diagram Γ on the source of an LTR L, and a configuration c of $\check{C}(L;\Gamma)$

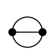
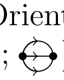
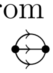
Γ: Order $E(\Gamma)$ arbitrarily, and order the half-edges as (First half-edge of the first edge, second half-edge of the first edge, ..., second half-edge of the last edge). The induced orientation of $H(\Gamma)$ is called the *edge-orientation* of $H(\Gamma)$. Note that it does not depend on the order of $E(\Gamma)$.

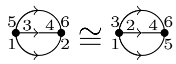
Lemma 2.6 *When Γ is equipped with a vertex-orientation, orientations of the manifold $\check{C}(L;\Gamma)$ are in canonical one-to-one correspondence with orientations of the set $H(\Gamma)$.*

PROOF: Since $\check{C}(L;\Gamma)$ is naturally an open submanifold of $\mathcal{L}^U \times \check{R}^T$, it inherits $\mathbb{R}^{\sharp U+3\sharp T}$-valued charts from $\mathbb{R}$-valued charts of $\mathcal{L}$ and $\mathbb{R}^3$-valued orientation-preserving charts of $\check{R}$. The $\mathbb{R}$-valued charts of $\mathcal{L}$ respect the local orientations of $\mathcal{L}$ induced by the corresponding oriented univalent vertices. In order to define the orientation of $\mathbb{R}^{\sharp U+3\sharp T}$, it suffices to identify its factors and order them (up to even permutation). Each of the factors may be labeled by an element of $H(\Gamma)$: the $\mathbb{R}$-valued local coordinate of an element of $\mathcal{L}$ corresponding to the image under j of an element u of U sits in the factor labeled by the half-edge that contains u; the 3 cyclically ordered (by the orientation of $\check{R}$) $\mathbb{R}$-valued local coordinates of the image under a configuration c of an element t of T live in the factors labeled by the three half-edges that contain t, which are cyclically ordered by the vertex-orientation of Γ, so that the cyclic orders match. □

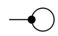
We use Lemma 2.6 to orient $\check{C}(R, L;\Gamma)$ as summarized in the following immediate corollary.

Corollary 2.7 *As soon as Γ is equipped with a vertex-orientation $o(\Gamma)$, if the edges of Γ are oriented, then the induced edge-orientation of $H(\Gamma)$ orients $\check{C}(L;\Gamma)$, via the canonical correspondence described in Lemma 2.6.*

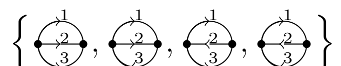
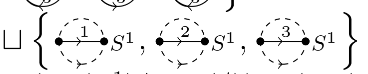
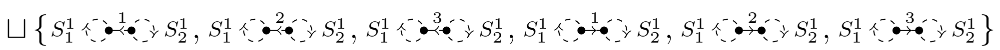
Example 2.8 Equip the diagram  with its vertex-orientation induced by the picture. Orient its three edges so that they start from the same vertex. Then the orientation of $\check{C}(R, L;$ $)$ induced by this edge-orientation of  matches the orientation of $(\check{R} \times \check{R}) \setminus \mathrm{diag}$ induced by the order of the two factors, where the first factor corresponds to the position of the vertex where the three edges start, as shown in the following picture.



For an integer $k \in \mathbb{N}$, set $\underline{k} = \{1, 2, \ldots, k\}$.

Definition 2.9 The *degree* of a Jacobi diagram is half the number of all its vertices. A *numbered degree n Jacobi diagram* is a degree n Jacobi diagram Γ whose edges are oriented, equipped with an injection $j_E \colon E(\Gamma) \hookrightarrow \underline{3n}$. Such an injection numbers the edges. Note that this injection is a bijection when $U(\Gamma)$ is empty. Let $\mathcal{D}^e_n(\mathcal{L})$ denote the set of numbered degree n Jacobi diagrams with support $\mathcal{L}$ without *looped edges* like .

Examples 2.10

$$\mathcal{D}^e_1(\emptyset) = $$

$$\mathcal{D}^e_1(S^1) = \mathcal{D}^e_1(\emptyset) \sqcup$$

$$\mathcal{D}^e_1(S^1_1 \sqcup S^1_2) = \mathcal{D}^e_1(\emptyset) \sqcup (\mathcal{D}^e_1(S^1_1) \setminus \mathcal{D}^e_1(\emptyset)) \sqcup (\mathcal{D}^e_1(S^1_2) \setminus \mathcal{D}^e_1(\emptyset))$$


Definition 2.11 Let Γ be a numbered degree n Jacobi diagram with support $\mathcal{L}$. An edge e oriented from a vertex v_1 to a vertex v_2 of Γ induces the following canonical map

$$\begin{array}{rccl} p_e \colon & \check{C}(R, L; \Gamma) & \to & C_2(R) \\ & c & \mapsto & (c(v_1), c(v_2)). \end{array}$$

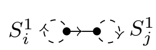
Let $o(\Gamma)$ be a vertex-orientation of Γ. For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(R), \tau)$. Define the *configuration space integral*

$$I\left(R, L, \Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}}\right) = \int_{(\check{C}(R,L;\Gamma), o(\Gamma))} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$$

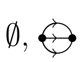
where $(\check{C}(R, L; \Gamma), o(\Gamma))$ denotes the manifold $\check{C}(R, L; \Gamma)$ equipped with the orientation induced by $o(\Gamma)$ and by the edge-orientation of Γ, as in Corollary 2.7.

Note that the dimension of the space $\check{C}(R, L; \Gamma)$ is equal to the degree of the integrated form $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$ since both coincide with the number of half-edges of Γ.

Examples 2.12 For any three propagating forms $\omega(1)$, $\omega(2)$ and $\omega(3)$ of $(C_2(R), \tau)$,

$$I(R, K_i \sqcup K_j \colon S^1_i \sqcup S^1_j \hookrightarrow \check{R}, \text{[diagram]}, (\omega(i))_{i \in \underline{3}}) = lk(K_i, K_j)$$


and

$$I(R, \emptyset, \text{[diagram]}, (\omega(i))_{i \in \underline{3}}) = \Theta(R, \tau)$$


for any numbering of the (plain) diagrams.

Definition 2.13 The involution $(x,y) \mapsto (y,x)$ of $\check{R}^2 \setminus \mathrm{diag}\left(\check{R}^2\right)$ extends to an involution ι $C_2(R)$. A propagating form ω of $(C_2(R),\tau)$ is *antisymmetric* if $\iota^*(\omega) = -\omega$. Let ι_{S^2} denote the antipodal map of S^2.

Since $\iota_{S^2}^*(\omega_{S^2}) = -\omega_{S^2}$, the standard propagating form $p_{S^2}^*(\omega_{S^2})$ of $(C_2(S^3),\tau_s)$ is antisymmetric. When the $\omega(i)$ are antisymmetric, $I\left(R,L,\Gamma,o(\Gamma),(\omega(i))_{i\in\underline{3n}}\right)$ is independent from the orientation of the edges of Γ. (Reversing the orientation of an edge changes the orientation of the configuration space and multiplies the integrated form by (-1).) For any propagating form ω of $(C_2(R),\tau)$, $\frac{1}{2}(\omega - \iota^*(\omega))$ is an antisymmetric propagating form ω of $(C_2(R),\tau)$.

When all the $\omega(i)$ coincide with a given propagating form ω, $I\left(R,L,\Gamma,o(\Gamma),(\omega(i))_{i\in\underline{3n}}\right)$ is simply denoted by $I\left(R,L,\Gamma,o(\Gamma),\omega\right)$. When $\check{R} = \mathbb{R}^3$, and when $\omega = p_{S^2}^*(\omega_{S^2})$, we simply write $I\left(L,\Gamma,o(\Gamma)\right)$ and we also drop $o(\Gamma)$ when Γ is oriented by a picture.

The study of these configuration space integrals was initiated by the articles of Witten [Wit89], Guadagnini, Martellini and Mintchev [GMM90], Bar-Natan [BN95b] on the perturbative expansion of the Chern-Simons theory,[10] in the case of links in $\mathbb{R}^3$, with the standard propagator $p_{S^2}^*(\omega_{S^2})$ on every edge. Let us compute some examples in this original setting.

2.2 Configuration space integrals associated to one chord

Let $K\colon S^1 \hookrightarrow \check{R}$ be a smooth embedding of the circle into $\check{R}$.

Consider the associated *configuration space*

$$\check{C}(K;\ominus) = \{(K(z), K(z\exp(2i\pi t))) \mid z \in S^1, t \in]0,1[\},$$

which is naturally identified with an open annulus $S^1\times]0,1[$, and set $I_\theta(K) = I\left(K,\ominus\right)$.

When $\check{R} = \mathbb{R}^3$, the *direction map*

$$\begin{array}{llll} d\colon & \check{C}(K;\Gamma) & \to & S^2 \\ & c & \mapsto & \frac{1}{\|K(z\exp(2i\pi t)) - K(z)\|}(K(z\exp(2i\pi t)) - K(z)) \end{array}$$

allows us to write

$$I_\theta(K) = I\left(K,\ominus\right) = \int_{\check{C}(K;\ominus)} d^*(\omega_{S^2}).$$

The annulus $\check{C}(K;\ominus)$ can be compactified to the closed annulus $C(K;\ominus) = S^1 \times [0,1]$, where d smoothly extends. The extended d, still denoted by d, maps $(z,0) \in S^1 \times \{0\}$ (resp. $(z,1) \in S^1 \times \{1\}$) to the direction of the tangent vector to K at z (resp. to the opposite direction).

In particular, our integral $I_\theta(K)$ converges. It is the *algebraic area* $\int_{d(C(K;\ominus))} \omega_{S^2}$ of $d(C(K;\ominus))$ in the following sense. The degree of d is a continuous map from $S^2 \setminus d(\partial C(K;\ominus))$ to $\mathbb{Z}$, and the

[10] The relation between the perturbative expansion of the Chern-Simons theory of the Witten article and the configuration space integral viewpoint is explained by Polyak in [Pol05] and by Sawon in [Saw06].

algebraic area of $d(C(K;\leftrightarrow))$ is $\int_{S^2} \deg(d)\omega_{S^2}$, which is the sum over the connected components C of $S^2 \setminus d(\partial C(K;\leftrightarrow))$ of the area of C multiplied by the value of the degree at C. Let us compute it for the following embeddings of the trivial knot.

Let O be an embedding of the circle to the horizontal plane. The image under d of the whole annulus is in the horizontal great circle of S^2. Its area is zero so that $I_\theta(O) = 0$

Let K_1 and K_{-1} be embeddings of S^1, which project to the horizontal plane as in Figure 10, which lie in the horizontal plane everywhere except when they cross over, and which lie in the union of two orthogonal planes.

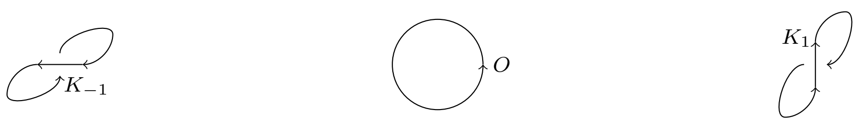
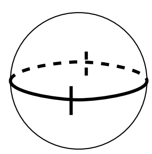


Figure 10: Diagrams of the trivial knot

The image of the boundary of $C(K;\leftrightarrow) = S^1 \times [0,1]$ in S^2 is in the union of the great circles of the two planes, or more precisely in the union of the horizontal plane and two vertical arcs as in the following figure.



In our example with K_1, the degree is constant on each side of our horizontal equator. We compute it at the North Pole $\vec{N}$ as in Subsection 1.1 and find that the degree of p_e is 1 on the Northern hemisphere. One similarly computes the degree of p_e on the Southern hemisphere, which is also 1.

Therefore, $I_\theta(K_1) = 1$. Similarly, $I_\theta(K_{-1}) = -1$.

Definition 2.14 A knot embedding K that lies in the union of the horizontal plane and a finite union of vertical planes so that the unit tangent vector to K is never vertical is called *almost horizontal*. An almost horizontal embedding K has a natural parallel $K_\parallel$ obtained from K by pushing it below. An embedding from S^1 to $\mathbb{R}^3$ is of *constant (resp. null) I_θ-degree* if the degree of the associated direction map ($d \colon \check{C}(K;\leftrightarrow) \to S^2$) can be extended to a constant (resp. everywhere 0) function on S^2.

Lemma 2.15 *Almost horizontal knot embeddings have constant I_θ-degree. Any knot of $\mathbb{R}^3$ may be represented by an almost horizontal knot embedding K. For an almost horizontal knot embedding K, $I_\theta(K) = lk(K, K_\parallel)$.*

PROOF: The *writhe* of an almost horizontal knot embedding is the number of positive crossings minus the number of negative crossings of its orthogonal projection onto the horizontal plane. As in the previously treated examples, we see that an almost horizontal knot embedding has a constant I_θ-degree, which is its *writhe*. The parallel below $K_\parallel$ is isotopic in the complement of K to the parallel $K_{\parallel,\ell}$ on the left-hand side of K, and the formulas of Section 1.1 show that $lk(K, K_{\parallel,\ell})$ is the writhe of K. □

It is easy to construct an embedding of null I_θ-degree in every isotopy class of embeddings of S^1 into $\mathbb{R}^3$, by adding kinks like or to a horizontal projection. Since I_θ continuously varies under an isotopy of K, for any knot K of $\mathbb{R}^3$, I_θ maps the space of embeddings of S^1 into $\mathbb{R}^3$ isotopic to K onto $\mathbb{R}$. Therefore, I_θ is not an isotopy invariant.

For a *long component* (i.e. a non-compact connected component) K of a long tangle representative in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, define

$$I_\theta(K) = 2I\left(K, \hat{\theta}, p^*_{S^2}(\omega_{S^2})\right) = 2I\left(K, \hat{\theta}\right).$$

Examples 2.16 Let us compute $I_\theta(K_{\ell,i}) = 2I\left(K_{\ell,i}, \hat{\theta}, \omega\right)$ for the long tangles of Figure 11, which shows their projections onto the plane $\mathbb{R} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{R}$. Assume that the images of the embeddings lie in this plane everywhere, except when they cross over so that the image of each one-component tangle is again in the union of two orthogonal planes.

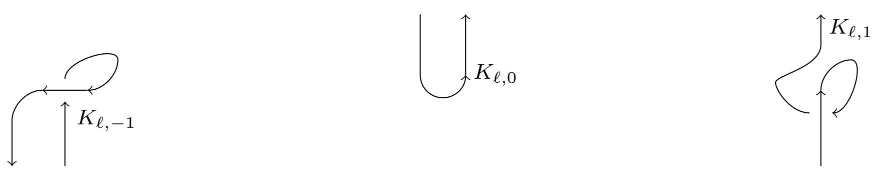


Figure 11: Long tangles representatives

The *configuration space* $\check{C}(K = K_{\ell,i}; \hat{\theta})$ associated to $\Gamma = \hat{\theta}$ and to $K\colon \mathbb{R} \hookrightarrow \mathbb{R}^3$ is

$$\check{C}(K; \hat{\theta}) = \{(K(t), K(u)) \mid (t,u) \in \mathbb{R}^2, t < u\},$$

which is naturally identified with the open triangle $\{(t,u) \in \mathbb{R}^2, t < u\}$. The *direction map*

$$\begin{array}{rlll} d\colon & \check{C}(K; \hat{\theta}) & \to & S^2 \\ & c & \mapsto & \frac{1}{\|K(u)-K(t)\|}(K(u) - K(t)) \end{array}$$

allows us to write

$$I_\theta(K) = 2I\left(K, \hat{\theta}\right) = 2\int_{\check{C}(K;\hat{\theta})} d^*(\omega_{S^2}).$$

Again, since $K_{\ell,0}$ is contained in $\mathbb{R} \times \mathbb{R}$, d maps $\check{C}(K_{\ell,0}; \hat{\theta})$ to the vertical great circle $S^1_{\mathbb{R}}$ that contains the real direction of $\mathbb{C}$ and $I_\theta(K_{\ell,0}) = 0$.

The configuration space $\check{C}(K_{\ell,1}; \hat{\Theta})$ embeds in the closed triangle

$$\tilde{C}(K_{\ell,1}; \hat{\Theta}) = \{(t,u) \in [-\infty,\infty]^2 \mid t \leq u\} = ◤ ,$$

where d extends. The extended d maps $(\{-\infty\} \times [-\infty,\infty]) \cup ([-\infty,\infty] \times \{\infty\})$ to the vertical upward vector $\vec{N}$, and it maps (u,u) to the unit tangent vector to K at u directed by $\mathbb{R}$. So far, this applies to any long K that goes from bottom to top. For our $K_{\ell,1}$, d maps the boundary of the triangle to the union of $S^1_{\mathbb{R}}$ and an arc of great circle. Here, the degree of d is 1 on the hemisphere behind $S^1_{\mathbb{R}}$ and it is zero in front of it so that $\int_{\check{C}(K_{\ell,1};\hat{\Theta})} d^*(\omega_{S^2}) = \frac{1}{2}$ and $I_\theta(K_{\ell,1}) = 1$.

Let us now compute $I_\theta(K_{\ell,-1}) = -1$. In this case, $\check{C}(K_{\ell,-1}; \hat{\Theta})$ still embeds into the former closed triangle but the map d does not continuously extend at $(-\infty,\infty)$. It extends to $\{-\infty\} \times [-\infty,\infty[$ and it maps $\{-\infty\} \times [-\infty,\infty[$ to $\vec{N}$, and it extends to $]-\infty,\infty] \times \{\infty\}$ and it maps $]-\infty,\infty] \times \{\infty\}$ to $(-\vec{N})$, but we need to blow-up the triangle at $(-\infty,\infty)$ so that d extends. After such a blow-up, which transforms the closed triangle into $\tilde{\tilde{C}}(K_{\ell,-1}; \hat{\Theta})$, (the extension of) d maps the boundary of $\tilde{\tilde{C}}(K_{\ell,-1}; \hat{\Theta})$ to the union of $S^1_{\mathbb{R}}$ and an arc of great circle. Here, the degree of d is -1 on the hemisphere in front of $S^1_{\mathbb{R}}$ and it is zero behind so that $I_\theta(K_{\ell,-1}) = -1$. (The closure $C(K_{\ell,-1}; \hat{\Theta})$ of $\check{C}(K_{\ell,-1}; \hat{\Theta})$ in $C_2(S^3)$ is a blow-up of $\tilde{\tilde{C}}(K_{\ell,-1}; \hat{\Theta})$.)

Definition 2.17 A propagating form of $(C_2(R),\tau)$ is *homogeneous* if its restriction to $\partial C_2(R)$ reads $p^*_\tau(\omega_{S^2})$ for the homogeneous volume one form ω_{S^2} of S^2.

Lemma 2.18 *Let $K \colon \mathbb{R} \hookrightarrow \check{R}$ be a component of a long tangle representative in an asymptotic rational homology $\mathbb{R}^3$. Let ω be a homogeneous propagating form of $(C_2(R),\tau)$. Then $I(R, K, \hat{\Theta}, \omega)$ is independent from the chosen homogeneous propagating form ω. (It depends on the embedding K and on τ.) It is denoted by $\frac{1}{2} I_\theta(K,\tau)$.*

See [Les20, Lemma 12.29].

2.3 More examples of configuration space integrals

Examples 2.19 For any trivalent numbered degree n Jacobi diagram

$$I(\Gamma) = I\left(S^3, \emptyset, \Gamma, o(\Gamma)\right) = 0.$$

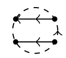
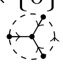
Indeed, $I(\Gamma)$ reads

$$\int_{(\check{C}(S^3,\emptyset;\Gamma),o(\Gamma))} \left(\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e\right)^* \left(\bigwedge_{e \in E(\Gamma)} \omega_{S^2}\right)$$

where

- $\bigwedge_{e \in E(\Gamma)} \omega_{S^2}$ is a product volume form of $(S^2)^{E(\Gamma)}$ with total volume one.

- $\check{C}(S^3, \emptyset; \Gamma)$ is the space $\check{C}_{\underline{3n}}(\mathbb{R}^3)$ of injections of $\underline{3n}$ into $\mathbb{R}^3$,
- the degree of $\wedge_{e \in E(\Gamma)} \omega_{S^2}$ is equal to the dimension of $\check{C}(S^3, \emptyset; \Gamma)$, and
- The map $\left(\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e\right)$ is never a local diffeomorphism since it is invariant under the action of global translations on $\check{C}(S^3, \emptyset; \Gamma)$.

Examples 2.20 Let us now compute $I\left(O, \Gamma, o(\Gamma), p^*_{S^2}(\omega_{S^2})\right)$ where O denotes the representative of the unknot of S^3, that is the image of the embedding of the unit circle S^1 of $\mathbb{C}$ regarded as $\mathbb{C} \times \{0\}$ into $\mathbb{R}^3$ regarded as $\mathbb{C} \times \mathbb{R}$ for the following graphs $\Gamma_1 =$ , $\Gamma_2 =$ , $\Gamma_3 =$ , $\Gamma_4 =$ . For $i \in \underline{4}$, set $I(\Gamma_i) = I\left(S^3, O, \Gamma_i, o(\Gamma_i), p^*_{S^2}(\omega_{S^2})\right)$. We are about to prove that $I(\Gamma_1) = I(\Gamma_2) = I(\Gamma_3) = 0$ and that $I(\Gamma_4) = \frac{1}{8}$.

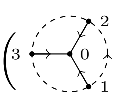
For $i \in \underline{4}$, set $\Gamma = \Gamma_i$, $I(\Gamma)$ again reads

$$\int_{(\check{C}(S^3, O; \Gamma), o(\Gamma))} \left(\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e\right)^* \left(\bigwedge_{e \in E(\Gamma)} \omega_{S^2}\right).$$

When $i \in \underline{2}$, the image of $\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e$ lies in the subset of $(S^2)^2$ made of the pair of horizontal vectors. Since the interior of this subset is empty, $I(\Gamma_i) = 0$. When $i = 3$, the two edges that have the same endpoints must have the same direction so that the image of $\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e$ lies in the subset of $(S^2)^{E(\Gamma)}$ where two S^2-coordinates are identical (namely the ones in the S^2-factors corresponding to the mentioned pair of edges), and $I(\Gamma_3) = 0$ as before.

Let us finish this series of examples by proving the following lemma.

Lemma 2.21 *Let $\Gamma = \Gamma_4$. Then*

$$I(\Gamma_4) = I\left(\right.$$ 
$$\left.\right) = I\left(S^3, O, \Gamma, o(\Gamma), p^*_{S^2}(\omega_{S^2})\right) = \frac{1}{8}.$$

PROOF: Let G^+ be the set of direct triples (X_{10}, X_{20}, X_{30}) of $(S^2)^3$ where all vectors have positive heights. Recall that ι_{S^2} is the antipodal map of S^2 and let $G^- = (\iota_{S^2})^E(G^+)$. Let D be the codimension one subspace of $(S^2)^3$ of triples of vectors such that a vector is horizontal or the three vectors are coplanar. For any edge e, let d_e denote $p_{S^2} \circ p_e$. It is easy to observe that the image of $\check{C}(K; \Gamma)$ under $\left(\prod_{e \in E} d_e\right)$ is contained in $G^+ \cup G^- \cup D$ and that the restriction of $\left(\prod_{e \in E} d_e\right)$ to the preimage of G^+ is a diffeomorphism h^+ onto G^+. Using the orientation-reversing diffeomorphism h_c of $\check{C}(K; \Gamma)$ that maps a configuration to its composition by $(-\mathrm{Id}_{\mathbb{R}^3})$ makes also clear that the restriction of $\left(\prod_{e \in E} d_e\right)$ to the preimage of G^- is the diffeomorphism $(\iota_{S^2})^E \circ h^+ \circ h_c$ onto G^-. In particular, the degree of $\left(\prod_{e \in E} d_e\right)$ is well-defined on $(S^2)^E \setminus D$, it is ± 1 on $G^+ \cup G^-$, with the same sign on G^+ and G^-, and 0 elsewhere. The sign is actually computed in the proof of [Les20, Lemma 7.11], the degree is 1 on G^+. This shows that $I_\Gamma(O)$ is twice the volume of G^+, so that $I_\Gamma(O) = \frac{1}{8}$. $\square$

2.4 More compactifications of configuration spaces

Axelrod, Singer [AS94] and Kontsevich [Kon94] proved that the configuration space integrals $I\left(R, L, \Gamma, o(\Gamma), (\omega(i))_{i\in \underline{3n}}\right)$ converge, when $\mathcal{L}$ is a disjoint union of circles using compactifications $C(R,L;\Gamma)$ " la Fulton-MacPherson" of $\check{C}(R,L;\Gamma)$ where the maps $p_e\colon \check{C}(R,L;\Gamma) \to C_2(R)$ smoothly extend so that $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ smoothly extends to $C(R,L;\Gamma)$, and

$$\int_{(\check{C}(R,L;\Gamma),o(\Gamma))} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e))) = \int_{(C(R,L;\Gamma),o(\Gamma))} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e))).$$

These compactifications are built as follows in [Les20, Chapter 8]. We first generalize the constructions of $C_2(R)$ and define a compactification $C_V(R)$ of the space $\check{C}_V(R)$ of injections of a finite set V into $\check{R}$ as in [Les20, Theorem 8.1] and as follows. For a non-empty $A \subseteq V$, let Ξ_A be the set of maps from V to R that map A to ∞ and $V \setminus A$ to $\check{R}$, injectively, and let $\mathrm{diag}_A(\check{R}^V)$ be the set of maps c from V to $\check{R}$, which are constant on A and which map $V \setminus A$ to $\check{R} \setminus \{c(A)\}$, injectively.

Start with R^V. Blow up Ξ_V (which is reduced to the point $m = \infty^V$ such that $m^{-1}(\infty) = V$). Then for $k = \sharp V, \sharp V - 1, \dots, 3, 2$, in this decreasing order, successively blow up the closures of the $\mathrm{diag}_A(\check{R}^V)$ such that $\sharp A = k$ (choosing an arbitrary order among them) and, next, the closures of the Ξ_J such that $\sharp J = k-1$ (again choosing an arbitrary order among them). Then the compactification $C(R,L;\Gamma)$ is the closure of $\check{C}(R,L;\Gamma)$ in $C_{V(\Gamma)}(R)$ as in [Les20, Proposition 8.3]. It satisfies the following properties.

Theorem 2.22 *If L is a link, then the configuration space $C(R,L;\Gamma)$ is a compact manifold with boundary and corners with the following properties.*

- *The interior of $C(R,L;\Gamma)$, which is the complement of $\partial C(R,L;\Gamma)$, is $\check{C}(R,L;\Gamma)$.*
- *For any edge e of Γ, the projection map $p_e\colon \check{C}(R,L;\Gamma) \to C_2(R)$ smoothly extends[11] to $C(R,L;\Gamma)$.*
- *For every non-empty subset A of $T(\Gamma)$, there is a codimension one open face $F_\infty(A,L,\Gamma)$ which reads as the product of*

$$\{c\colon (V\setminus A) \hookrightarrow \check{R} \mid c_{|U} = L \circ j_\Gamma(c) \text{ for some } j_\Gamma(c) \in [i_\Gamma]\}$$

  by the space $\check{\mathcal{S}}(\mathbb{R}^3, A)$ of injective *maps w from A to $(\mathbb{R}^3 \setminus 0)$ up to dilation[12]. An element $(c,[w])$ of this face is the limit in $C(R,L;\Gamma)$ when u approaches 0 of a family of injective configurations $(c, \frac{1}{u}w)_{u\in]0,\varepsilon[}$, which is defined for some small $\varepsilon > 0$, and for a representative w of $[w]$.*

[11] See [Les20, Theorem 8.2].

[12] Dilations are homotheties with positive ratio.

- *For every subset A of cardinality greater than 2 of $V(\Gamma)$ that intersects $U = U(\Gamma)$ as a (possibly empty) set of consecutive vertices on some component of $\mathcal{L}$ with respect to $[i_\Gamma]$, there is a codimension one open face $F(A, L, \Gamma)$ which behaves as follows. Let $a \in A$ be such that $a \in A \cap U$ if $A \cap U \neq \emptyset$. Then $F(A, L, \Gamma)$ fibers over*

$$\{c \colon (V \setminus A) \cup \{a\} \hookrightarrow \check{R} \mid c_{|(U \setminus (U \cap (A \setminus \{a\}))} = L \circ j_\Gamma(c)_{|(U \setminus (U \cap (A \setminus \{a\}))} \text{ for some } j_\Gamma(c) \in [i_\Gamma]\}.$$

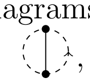
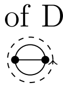
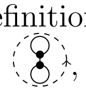
  - *If $A \cap U = \emptyset$, then the fiber is the space $\check{\mathcal{S}}_A(T_{c(a)}\check{R})$ made of* injective *maps w_A from A to $T_{c(a)}\check{R}$ up to translation and dilation. When $\check{R} = \mathbb{R}^3$, an element $(c, [w_A])$ of this face is the limit in $C(R, L; \Gamma)$ when u approaches 0 of a family of injective configurations $(c + uw_A)_{u \in]0,\varepsilon[}$, which is defined for some small $\varepsilon > 0$, where w_A is a representative of w_A which maps a to zero, and c and w_A are extended to V so that c is constant on A and w_A maps $V \setminus A$ to 0.*
  - *If $A \cap U \neq \emptyset$, then the fiber over c is the space of injective maps w_A from A to $T_{c(a)}\check{R}$ which map $A \cap U$ to the line $\mathbb{R}TL_{c(a)}$ through 0 directed by the tangent vector $TL_{c(a)}$ to L at $c(a)$, with respect to an order compatible with i_Γ, up to dilation and translation along the line $\mathbb{R}TL_{c(a)}$.*

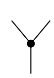
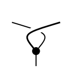
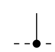
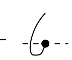
- *The complement of the union of the faces described above in the boundary of $C(R, L; \Gamma)$ is a finite union of manifolds of codimension at least 2 in $C(R, L; \Gamma)$.*

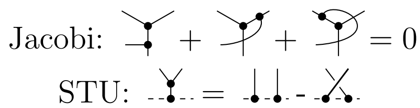
These faces are described more precisely in [Les20, Section 8.3]. Bott and Taubes analyzed the variations of the integrals $I_\Gamma(K)$ when a knot K of $\mathbb{R}^3$ varies in its isotopy class in [BT94], using such compactifications together with their codimension one faces, described above, which correspond to the loci where one blow-up has been performed.

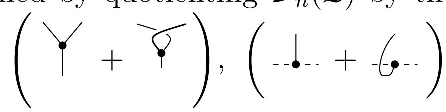
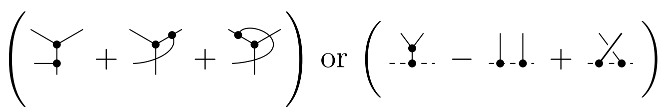
In [Poi00], Sylvain Poirier used the theory of semi-algebraic sets [BCR98] to prove the convergence of the integrals for semi-algebraic long tangle representatives in $\mathbb{R}^3$. He proved that the closure $C(L; \Gamma)$ of $\check{C}(L; \Gamma)$ in $C_{V(\Gamma)}(S^3)$ is a semi-algebraic set for semi-algebraic long tangle representatives L in $\mathbb{R}^3$. In [Les20, Chapter 13], I proved the convergence of the integrals for all long tangle representatives L in $\mathbb{Q}$–spheres by studying the structure of the closure of $\check{C}(R, L; \Gamma)$ in $C_{V(\Gamma)}(R)$. This closure is not a manifold anymore. See [Les20, Lemma 13.23].

2.5 The invariant Z

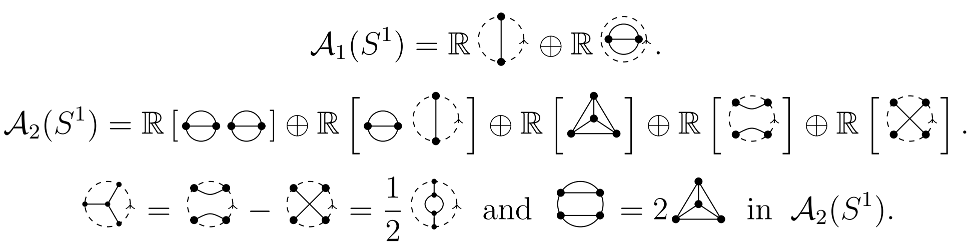
For a one-manifold $\mathcal{L}$, $\mathcal{D}_n(\mathcal{L})$ denotes the real vector space generated by the degree n oriented Jacobi diagrams on $\mathcal{L}$ of Definition 2.2. For the circle S^1, these generators of $\mathcal{D}_1(S^1)$ are the diagrams , , , , and the diagrams obtained from them by changing some vertex orientations. For a non-necessarily oriented one-manifold $\mathcal{L}$, $\mathcal{A}_n(\mathcal{L})$ denotes the quotient of $\mathcal{D}_n(\mathcal{L})$ by the following relations AS, Jacobi and STU:

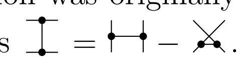
AS (or antisymmetry):  +  $= 0$ and  +  $= 0$



Each of these relations relate oriented Jacobi diagrams which are identical outside the pictures (or, more exactly, which can be represented by planar immersions whose images intersect a disk as in the picture and are identical outside this disk). The quotient $\mathcal{A}_n(\mathcal{L})$ is the largest quotient of $\mathcal{D}_n(\mathcal{L})$ in which these relations hold. It is obtained by quotienting $\mathcal{D}_n(\mathcal{L})$ by the vector space generated by elements of $\mathcal{D}_n(\mathcal{L})$ of the form ,  or.

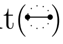
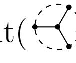
Examples 2.23 Note that diagrams with looped edges vanish in $\mathcal{A}_n(\mathcal{L})$.



Remark 2.24 When $\partial\mathcal{L} = \emptyset$, Lie algebras provide nontrivial linear maps, called *weight systems* from $\mathcal{A}_n(\mathcal{L})$ to $\mathbb{K}$, see [BN95a], [CDM12, Chapter 6] or [Les05, Section 6]. In the weight system constructions, the Jacobi relation for the Lie bracket ensures that the maps defined for oriented Jacobi diagrams factor through the Jacobi relation. In [Vog11], Pierre Vogel proved that the maps associated with Lie (super)algebras are sufficient to detect nontrivial elements of $\mathcal{A}_n(\mathcal{L})$ until degree 15, and he exhibited a non trivial element of $\mathcal{A}_{16}(\emptyset)$ that cannot be detected by such maps. The Jacobi relation was originally called IHX by Bar-Natan in [BN95a] because, up to AS, it can be written as . Note that the four entries in this IHX relation play the same role, up to AS.

Let $\mathcal{D}^u_n(\mathcal{L})$ denote the set of unnumbered, unoriented degree n Jacobi diagrams on $\mathcal{L}$ without looped edges. Note that the product $I(R, L, \Gamma, \omega)[\Gamma]$ is independent of the orientation of Γ for an antisymmetric propagating form of $(C_2(R), \tau)$.

An *automorphism* of a graph $\Gamma \in D^u_n(\mathcal{L})$ is an automorphism of the underlying uni-trivalent graph, for which the permutation σ of $U(\Gamma)$ induced by the automorphism is such that $i_\Gamma \circ \sigma$ and i_Γ are isotopic for some (and then any) Γ-compatible injection i_Γ. Let $\mathrm{Aut}(\Gamma)$ denote the set of these automorphisms, and let $\sharp\mathrm{Aut}(\Gamma)$ denote its cardinality.

Examples 2.25 The cardinality of Aut() is 2, $\sharp$Aut() $= 1$, $\sharp$Aut() $= 12$, $\sharp$Aut() $= 3$.

The following theorem is a consequence of [Les20, Theorem 7.20 and Proposition 10.6], when L is a link and of [Les20, Theorem 12.32, Proposition 12.36 and Lemma 12.38] in general.

Theorem 2.26 *Let L be a long tangle representative in $\check{R}$. Let L_C denote the set of connected components of L. Let ω be an antisymmetric homogeneous propagating form of $(C_2(R), \tau)$. Then*

$$Z_n(\check{R}, L, (I_\theta(K,\tau))_{K \in L_C}, p_1(\tau)) = \sum_{\Gamma \in \mathcal{D}_n^u(\mathcal{L})} \frac{1}{\sharp Aut(\Gamma)} I(R, L, \Gamma, \omega)[\Gamma] \in \mathcal{A}_n(\mathcal{L})$$

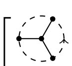
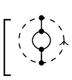
only depends on

- *the pair $(\mathcal{C}, L \cap \mathcal{C})$ up to orientation-preserving diffeomorphisms[13] of $\mathcal{C}$ which preserve the bottom disk $D^2 \times \{0\}$ and the top disk $D^2 \times \{1\}$, and which preserve $c^+(B^+)$ and $c^-(B^-)$ up to (global) translation and dilation,*
- *$I_\theta(K,\tau)$ for each component K of L,*
- *$p_1(\tau)$,*

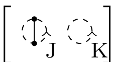
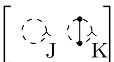
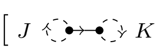
where $I(R, L, \Gamma, \omega)[\Gamma] = I(R, L, \Gamma, o(\Gamma), \omega)[\Gamma, o(\Gamma)]$ for an arbitrary orientation of Γ.

Note that the above definition of $I(R, L, \Gamma, \omega)[\Gamma]$ is consistent because the right-hand side of the above equality does not depend on $o(\Gamma)$. Also note that when L is an almost horizontal knot K of $\mathbb{R}^3$ as in Definition 2.14, $Z_n(\mathbb{R}^3, K, I_\theta(K,\tau_s), p_1(\tau_s) = 0)$ only depends on $I_\theta(K,\tau_s) = lk(K, K_\parallel)$ (see Lemma 2.15) and on the isotopy class of K, so that Z_n provides an isotopy invariant of parallelized knots in $\mathbb{R}^3$.

Examples 2.27 For the empty link $\emptyset$ of $\mathbb{R}^3$, $Z_n(\mathbb{R}^3, \emptyset, 0) = 0$ for all $n > 0$ and $Z_0(\mathbb{R}^3, \emptyset, 0) = [\emptyset]$. For the knot O of Example 2.20, $Z_0(\mathbb{R}^3, O, 0) = [$  $]$, $Z_1(\mathbb{R}^3, O, 0) = 0$ and

$Z_2(\mathbb{R}^3, O, 0) = \frac{1}{24}\Big[$  $\Big] = \frac{1}{48}\Big[$  $\Big]$.

For any two-component link $J \sqcup K$ of $\mathbb{R}^3$ such that J and K are almost horizontal,

$Z_1(\mathbb{R}^3, J \sqcup K, 0) = \frac{1}{2} lk(J, J_\parallel)\Big[$  $\Big] + \frac{1}{2} lk(K, K_\parallel)\Big[$  $\Big] + lk(J,K)\,[$  $]$.

If $(\check{R}, \tau)$ is a parallelized asymptotic rational homology $\mathbb{R}^3$, then

$Z_1(\check{R}, \emptyset, p_1(\tau)) = \frac{\Theta(R,\tau)}{12}\,[$  $]$.

[13] As often in these notes, we identify an embedding and its image.

Remark 2.28 Let ω be an antisymmetric homogeneous propagating form of $(C_2(R), \tau)$. The homogeneous definition of $Z_n(\check{R}, L, .)$ above makes clear that $Z_n(\check{R}, L, .)$ is a measure of graph configurations where a graph configuration is an embedding of the set of vertices of a uni-trivalent graph into $\check{R}$, which maps univalent vertices to $\mathcal{L}(L)$ in a constrained way. The embedded vertices are connected by a set of abstract plain edges, which represent the measuring form. The factor $\frac{1}{\sharp \mathrm{Aut}(\Gamma)}$ ensures that every such configuration of an unnumbered, unoriented graph is measured once.

Definition 2.29 A long tangle representative $L\colon \mathcal{L} \hookrightarrow \check{R}$ is *straight* (with respect to τ)[14] if p_τ maps the unit bundles to the circle components of L and the unit bundles to the components of L from bottom to top and from top to bottom to $\{\pm\vec{N}\}$, and if p_τ maps the boundary $\partial C(K; \hat{\circlearrowleft})$ of the closure $C(K; \hat{\circlearrowleft})$ of $\check{C}(K; \hat{\circlearrowleft})$ in $C_2(R)$ to a half-circle in the great circle $S^1_{\mathbb{R}}$ of S^2 in the plane $(\mathbb{R} \times \mathbb{R}) \subset (\mathbb{C} \times \mathbb{R})$, for every connected component K of L.

Let B_3^c be the closure of the complement of the compact ball B_3 of radius 3 centered at 0 in $\mathbb{R}^3$. Every component K of a straight long tangle representative has a natural parallel $K_\parallel$ up to isotopy, which fixes B_3^c. This parallel is obtained by translating K by $(z, 0) \in \mathbb{C} \times \mathbb{R}$ on B_3^c, and by pushing K in the direction of $\tau(., z)$ without crossing (other parts of) K in B_3, where $z = \frac{i}{100}$ for components that go from bottom to top or from top to bottom, and $z = \frac{-1}{100}$ for components that go from bottom to bottom or from top to top. Generalize the *linking number* $lk(K, K_\parallel)$ to long components by closing K (resp. $K_\parallel$) to an embedded circle $\hat{K}$ (resp. $\hat{K}_\parallel$) obtained by replacing the two half-lines of $K \cap B_3^c$ (resp. of $K_\parallel \cap B_3^c$) with a single arc in ∂B_3 that connects the two points of $K \cap \partial B_3$ (resp. of $K_\parallel \cap \partial B_3$) as explained below and by setting $lk(K, K_\parallel) = lk(\hat{K}, \hat{K}_\parallel)$. For a component that goes from bottom to top or from top to bottom, choose the connecting arc of $\hat{K}$ in a great circle of ∂B_3 on the left-hand side, and the connecting arc of $\hat{K}_\parallel$ in a great circle of ∂B_3 on the right-hand side. For other long components, choose the connecting arcs of $\hat{K}$ and $\hat{K}_\parallel$ in the shortest possible arcs of great circle of ∂B_3 that contain their given ends.

Note that for any long tangle representative L in $\check{R}$, and for any asymptotically standard parallelization τ_0 of $\check{R}$, there is a parallelization τ homotopic to τ_0 among asymptotically standard parallelizations τ_0 of $\check{R}$ such that L is straight with respect to τ.

Lemma 2.30 *For any component K of a straight embedding, $I_\theta(K, \tau) = lk(K, K_\parallel)$.*

PROOF: Note that both sides of the equality to be proved are independent of the orientation of K. Therefore, [Les20, Proposition 15.6] reduces the proof of the lemma to the proof that the above definition of $lk(K, K_\parallel)$ coincides with [Les20, Definition 12.15], for some orientation of

[14] Our definition of straight tangles differs from [Les20, Definition 15.5]. On one hand, it is more restrictive because it implies that the bottom (resp.top) configuration of a configuration that goes from bottom to bottom (resp. from top to top) lies in a line parallel to the real line. On the other hand, p_τ may map the unit bundle to any half-circle of $S^1_{\mathbb{R}}$, here.

K. This is immediate if $\partial C(\pm K;$  $)$ is mapped to the half-circle from $-\vec{N}$ to $\vec{N}$ that contains $1 \in \mathbb{C}$, and if $\pm K$ goes from bottom to top. The other cases are also easy. $\square$

For a degree n Jacobi diagram Γ on $\mathcal{L}$, set

$$\zeta_\Gamma = \frac{(3n - \sharp E(\Gamma))!}{(3n)! 2^{\sharp E(\Gamma)}}.$$

Theorem 2.31 *Let $L\colon \mathcal{L} \hookrightarrow \check{R}$ be a straight embedding of a disjoint union of circles with respect to τ in $(\check{R}, \tau)$, which is our asymptotic rational homology $\mathbb{R}^3$. For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(R), \tau)$, and let P_i be a propagating chain of $(C_2(R), \tau)$. With the notations of Definition 2.9 and Theorem 2.26,*

$$\begin{aligned} Z_n(\check{R}, L, (lk(K, K_\parallel))_{K \in L_C}, p_1(\tau)) &= \textstyle\sum_{\Gamma \in \mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i \in \underline{3n}})[\Gamma] \\ &= \textstyle\sum_{\Gamma \in \mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma [\bigcap_{e \in E(\Gamma)} p_e^{-1}(P_{j_E(e)})][\Gamma]. \end{aligned}$$

whenever all the above intersections are transverse, and they are for generic choices of $(P_i)_{i \in \underline{3n}}$. In particular, the right-hand sides do not depend on our choices and they are rational.

Proof: The first equality is a consequence of [Les20, Theorem 7.29]. The genericity of the statement is described in [Les20, Chapter 11]. See [Les20, Definition 11.3 and Lemma 11.4], in particular. The second equality is a consequence of [Les20, Lemma 11.7]. $\square$

In the statement above, $[\bigcap_{e \in E(\Gamma)} p_e^{-1}(P_{j_E(e)})]$ is the algebraic intersection of the codimension 2 chains $p_e^{-1}(P_{j_E(e)})$ in $C(R, L; \Gamma)$. Theorem 2.31 may be applied to compute Z with the Morse propagators of Section 1.5. In this case Z counts graph embeddings where some edges embed in the flow lines (when the pairs of points are in the part P_ϕ of $P(f, \mathfrak{g})$) and some edges $e = (v, w)$ constrain their origin vertex to live in some descending manifold $\mathcal{B}_j$ of an index 2 critical point and their final vertex to live in some ascending manifold $\mathcal{A}_i$ of an index 1 critical point, up to some corrections due to the behaviour of $P(f, \mathfrak{g})$ near $\partial C_2(R)$. Such a way of counting graphs had been first proposed by Fukaya in [Fuk96] and further studied by Watanabe [Wat18].

The following result similar to Theorem 2.31 can be deduced from independent results of Sylvain Poirier [Poi02] and Dylan Thurston [Thu99] in the case of links in $\mathbb{R}^3$, with propagating chains $p_{S^2}^{-1}(X_i)$. Recall that d_e denotes $p_{S^2} \circ p_e$ for an edge e. Also recall Definition 2.14 of knots of constant I_θ-degree, and Lemma 2.15, which implies that for almost horizontal knot embeddings K, $I_\theta(K) = lk(K, K_\parallel)$.

Theorem 2.32 *Let $L\colon \mathcal{L} \hookrightarrow \mathbb{R}^3$ be an embedding of a disjoint union of circles into $\mathbb{R}^3$ such that all components of L have a constant I_θ-degree. The subset A of $(S^2)^{3n}$ made of the $(X_i)_{i \in \underline{3n}}$ such that $(X_{j_E(e)})_{e \in E(\Gamma)}$ is a regular value of $\prod_{e \in E(\Gamma)} d_e\colon C(L; \Gamma) \to (S^2)^{j_E(E(\Gamma))}$ for any $\Gamma \in \mathcal{D}^e_n(\mathcal{L})$ is open and dense, and, for any $(X_i)_{i \in \underline{3n}} \in A$,*

$$Z_n(\mathbb{R}^3, L, (I_\theta(K))_{K \in L_C}, 0) = \textstyle\sum_{\Gamma \in \mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma [\bigcap_{e \in E(\Gamma)} d_e^{-1}(X_{j_E(e)})][\Gamma].$$

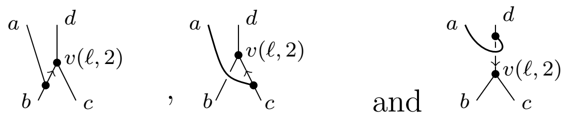


Figure 12: Around a collapsing edge

This theorem tells us that $Z_n(\mathbb{R}^3, L, (I_\theta(K))_{K\in L_C}, 0)$ behaves as an $\mathcal{A}_n(\mathcal{L})$–valued degree on $(S^2)^{3n}$ and it may be proved along these lines. Associate the map $\Pi_\Gamma = \prod_{e\in E(\Gamma)} d_e \times \mathrm{Id}_{(S^2)^{\underline{3n}\setminus j_E(E(\Gamma))}}$ from $C(L;\Gamma) \times (S^2)^{\underline{3n}\setminus j_E(E(\Gamma))}$ to $(S^2)^{3n}$ to each $\Gamma \in \mathcal{D}^e_n(\mathcal{L})$, equipped with a fixed arbitrary orientation, here. By definition, for any such Jacobi diagram Γ equipped with an implicit vertex-orientation,

$$I\left(L, \Gamma, p^*_{S^2}(\omega_{S^2})\right) = \int_{\check{C}(L;\Gamma)} \bigwedge_{e\in E(\Gamma)} d^*_e(\omega_{S^2})$$

is the algebraic volume of the image of Π_Γ. The degree d_Γ of Π_Γ is a continuous function on the complement of $\Pi_\Gamma\left(\partial C(L;\Gamma) \times (S^2)^{\underline{3n}\setminus j_E(E(\Gamma))}\right)$ in $(S^2)^{3n}$. The degree d_Γ jumps by ± 1 across each wall, where a *wall* is a codimension one image of a codimension one face of $\Pi_\Gamma\left(C(L;\Gamma) \times (S^2)^{\underline{3n}\setminus j_E(E(\Gamma))}\right)$. Sylvain Poirier and Dylan Thurston independently proved that $D_n = \sum_{\Gamma\in\mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma d_\Gamma[\Gamma]$ can be extended to an $\mathcal{A}_n(\mathcal{L})$-valued constant function on $(S^2)^{3n}$ by gluing the above occuring walls as in the example of a *Jacobi gluing* discussed below.

Let $\Gamma \in \mathcal{D}^e_n(\mathcal{L})$. Let $e(\ell)$ be an edge of Γ with label ℓ, which goes from a vertex $v(\ell, 1)$ to a vertex $v(\ell, 2)$. Assume that no other edge of Γ contains both $v(\ell, 1)$ and $v(\ell, 2)$. Let $\Gamma/e(\ell)$ be the labelled edge-oriented graph obtained from Γ by contracting $e(\ell)$ to one point. (The labels of the edges of $\Gamma/e(\ell)$ belong to $\underline{3n} \setminus \{\ell\}$, $\Gamma/e(\ell)$ has one four-valent vertex and its other vertices are univalent or trivalent.) Let $\mathcal{E} = \mathcal{E}(\Gamma; e(\ell))$ be the set of pairs $(\tilde\Gamma, \tilde e(\ell))$ where $\tilde\Gamma \in \mathcal{D}^e_n(\mathcal{L})$ and $\tilde e(\ell)$ is an edge of $\tilde\Gamma$ with label ℓ such that $\tilde\Gamma/\tilde e(\ell)$ is equal to $\Gamma/e(\ell)$.

Let us show that there are 6 graphs in $\mathcal{E}$. Let a, b, c, d be the four half-edges of $\Gamma/e(\ell)$ that contain its four-valent vertex. In $\tilde\Gamma$, Edge $\tilde e(\ell)$ goes from a vertex $v(\ell, 1)$ to a vertex $v(\ell, 2)$. Vertex $v(\ell, 1)$ is adjacent to the first half-edge of $\tilde e(\ell)$ and to two half-edges of $\{a, b, c, d\}$. The unordered pair of $\{a, b, c, d\}$ adjacent to $v(\ell, 1)$ determines $\tilde\Gamma$ as an element of $\mathcal{D}^e_n(\mathcal{L})$ and there are 6 elements in $\mathcal{E}$ labelled by the pairs of elements of $\{a, b, c, d\}$. They are $\Gamma = \Gamma_{ab}$, Γ_{ac}, Γ_{ad}, Γ_{bc}, Γ_{bd} and Γ_{cd}, equipped with the edge from $v(\ell, 1)$ to $v(\ell, 2)$. Three of them (Γ_{ab}, Γ_{ac} and Γ_{ad}) are drawn in Figure 12. The other ones are obtained from them by reversing the orientation of $\tilde e(\ell)$.

The face $F(A, L, \Gamma)$ is fibered over the configuration space of $\Gamma/e(\ell)$ with fiber S^2, which contains the (free) direction of the vector from $c(v(\ell, 1))$ to $c(v(\ell, 2))$, so that the wall determined by this space is the same for all $(\tilde\Gamma, \tilde e(\ell))$ in $\mathcal{E}$, while the jump of D_n across the wall associated to $(\tilde\Gamma, \tilde e(\ell))$ is $\pm\zeta_\Gamma[\tilde\Gamma]$. Checking the signs as in [Les20, Lemma 9.12] shows that the sum over the elements of $\mathcal{E}$ of the jumps of D_n across the wall associated to $(\tilde\Gamma, \tilde e(\ell))$ vanishes, thanks to the Jacobi relation.

3 Some properties of Z

Set $\mathcal{A}(\mathcal{L}) = \prod_{n\in\mathbb{N}} \mathcal{A}_n(\mathcal{L})$. We drop the subscript n to denote the collection (or the sum) of the Z_n for $n \in \mathbb{N}$. For example,

$$Z\left(\check{R}, L, (0), p_1(\tau)\right) = \left(Z_n(\check{R}, L, (0), p_1(\tau))\right)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} Z_n\left(\check{R}, L, (0), p_1(\tau)\right) \in \mathcal{A}(\mathcal{L}).$$

The disjoint union of diagrams induces a commutative product on $\mathcal{A}(\emptyset)$ which maps two classes of diagrams to the class of their disjoint union. Equipped with this product, $\mathcal{A}(\emptyset)$ is a commutative algebra. The disjoint union of diagrams similarly induces an $\mathcal{A}(\emptyset)$-module structure on $\mathcal{A}(\mathcal{L})$ for any one-manifold $\mathcal{L}$.

3.1 On the invariant Z of $\mathbb{Q}$–spheres and the anomaly β

Let $\mathcal{A}^c_n(\emptyset)$ denote the subspace of $\mathcal{A}_n(\emptyset)$ generated by trivalent Jacobi diagrams with one connected component, set $\mathcal{A}^c(\emptyset) = \prod_{n\in\mathbb{N}} \mathcal{A}^c_n(\emptyset)$ and let $p^c \colon \mathcal{A}(\emptyset) \to \mathcal{A}^c(\emptyset)$ be the linear projection that maps the empty diagram and diagrams with several connected components to 0. Let $\mathcal{D}^c_n$ denote the subset of $\mathcal{D}^e_n(\emptyset)$ that contains the connected diagrams of $\mathcal{D}^e_n(\emptyset)$. For $n \in \mathbb{N}$, set

$$z_n(\check{R}, p_1(\tau)) = p^c\left(Z_n(\check{R}, p_1(\tau))\right).$$

$$z_n(\check{R}, p_1(\tau)) = \sum_{\Gamma\in\mathcal{D}^c_n} \zeta_\Gamma I(R, \Gamma, \omega)[\Gamma] \in \mathcal{A}^c_n(\emptyset)$$

for some propagating form ω of $(C_2(R), \tau)$. The reader can check that

$$Z(\check{R}, p_1(\tau)) = \exp\left(z(\check{R}, p_1(\tau))\right).$$

The dependence on $p_1(\tau)$ of $z(\check{R}, p_1(\tau))$ is linear and the following proposition is shown in [Les20, Propositions 10.13 and 10.14].

Proposition 3.1 (Kuperberg, Thurston [KT99]) *There exists an element $\beta \in \mathcal{A}(\emptyset)$ such that $\left(z(\check{R}, p_1(\tau)) - \frac{p_1(\tau)}{4}\beta\right)$ is independent of τ so that*

$$Z(R) = Z(\check{R}, p_1(\tau)) \exp\left(-\frac{p_1(\tau)}{4}\beta\right).$$

is an invariant of R. The degree n part β_n of $\beta = (\beta_n)_{n\in\mathbb{N}}$ is zero, when n is even.

Note the following proposition.

Proposition 3.2 *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$, then*

$$Z_1(\check{R}, p_1(\tau)) = z_1(\check{R}, p_1(\tau)) = \frac{\Theta(R,\tau)}{12}[\ominus]$$

in $\mathcal{A}_1(\emptyset) = \mathcal{A}_1(\emptyset; \mathbb{R}) = \mathbb{R}[\ominus]$.

In particular, $\beta_1 = \frac{1}{12}[\ominus]$. See [Les20, Section 10.2] for more details about the anomaly β, which is unknown in odd degrees greater than 1.

In [KT99], Greg Kuperberg and Dylan Thurston proved that the restriction of Z to $\mathbb{Z}$-spheres is a universal finite invariant of $\mathbb{Z}$-spheres, with respect to the Ohtsuki theory of finite type invariants for $\mathbb{Z}$-spheres [Oht96], see also [GGP01]. In [Les04], I generalized their result by proving that the restriction of Z to $\mathbb{Q}$-spheres is a universal finite invariant of $\mathbb{Q}$-spheres with respect to the Moussard theory of finite invariant of $\mathbb{Q}$-spheres based on Lagrangian-preserving surgeries [Mou12], see [Les20, Section 17.1]. This implies that Z and the LMO invariant of Le, Murakami and Ohtsuki [LMO98] are equivalent in the sense that they distinguish the same $\mathbb{Q}$–spheres.

3.2 On the invariant Z of framed tangles and the anomaly α

The product $\exp\left(-\frac{p_1(\tau)}{4}\beta\right) Z(\check{R}, L, (I_\theta(K,\tau))_{K \in L_C}, p_1(\tau))$ is actually independent of $p_1(\tau)$, too, so that we drop $p_1(\tau))$ and set

$$Z(\check{R}, L, (I_\theta(K,\tau))_{K \in L_C}) = \exp\left(-\frac{p_1(\tau)}{4}\beta\right) Z(\check{R}, L, (I_\theta(K,\tau))_{K \in L_C}, p_1(\tau)).$$

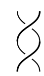
Remark 3.3 Let $\check{\mathcal{A}}(\mathcal{L})$ be the quotient of $\mathcal{A}_n(\mathcal{L})$ by the vector space generated by the diagrams that have at least one connected component without univalent vertices. Using the corresponding projection $\check{p}\colon \mathcal{A}(\mathcal{L}) \to \check{\mathcal{A}}(\mathcal{L})$ and setting $\check{Z}_n = \check{p} \circ Z_n$, we can write

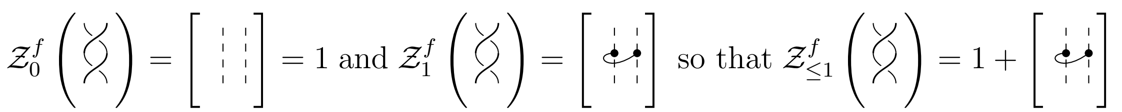
$$Z(\check{R}, L, (I_\theta(K,\tau))_{K \in L_C}, p_1(\tau)) = Z(\check{R}, p_1(\tau))\check{Z}(\check{R}, L, (I_\theta(K,\tau))_{K \in L_C}).$$

As soon as a one manifold $\mathcal{L}$ reads as the union of two one-manifolds $\mathcal{L}_1$ and $\mathcal{L}_2$, which only meet along their boundaries, the disjoint union of diagrams again induces products from $\mathcal{A}_j(\mathcal{L}_1) \otimes \mathcal{A}_k(\mathcal{L}_2)$ to $\mathcal{A}_{j+k}(\mathcal{L})$ where the needed class of injections $i_{\Gamma_1 \sqcup \Gamma_2}$ for a disjoint union of a Jacobi diagram Γ_1 on $\mathcal{L}_1$ and a Jacobi diagram Γ_2 on $\mathcal{L}_2$ is naturally induced by $[i_{\Gamma_1}]$ and $[i_{\Gamma_2}]$. View $[0,1]$ as the union of $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, together with orientation-preserving identifications of $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ with $[0,1]$. Then the above products induce an algebra structure on $\mathcal{A}([0,1])$. In [BN95a], Bar-Natan proved that the induced product of $\mathcal{A}([0,1])$ is actually commutative, and that the natural map from $\mathcal{A}([0,1])$ to $\mathcal{A}(S^1)$ obtained from the identification $S^1 = [0,1]/0 \sim 1$ is an ismorphism. See [Les20, Proposition 6.21]. In particular, the choice of an oriented connected component $\mathcal{K}$ of $\mathcal{L}$ equips $\mathcal{A}(\mathcal{L})$ with a well-defined $\mathcal{A}([0,1])$-*module structure* $\sharp_{\mathcal{K}}$, induced by an orientation-preserving inclusion from $[0,1]$ into a little part of $\mathcal{K}$ outside the vertices.

A *tangle representative* is a pair $(\mathcal{C}, \mathcal{C} \cap L)$, which is simply denoted by $(\mathcal{C}, L)$ for a long tangle representative as in Definition 2.4, where we again identify the embedding L and its image. Such a tangle representative is a cobordism in $\mathcal{C}$ from the bottom configuration of L to the top configuration of L. From now on Z is seen as a map, which maps such a tangle representative, still denoted by L or by $(\mathcal{C}, L)$, to an element of $\mathcal{A}(\mathcal{L})$.

Note that Z maps trivial braids $c(B) \times [0,1]$ of $\mathcal{C}_0$ to the class of the empty diagram, since the vertical translations act on the involved configuration spaces so that the image of $\prod_{e \in E} d_e$ in $(S^2)^{j_E(E)}$ of the configuration space is the image of the quotient, which lives in a subspace of codimension at least 1.

It is easy to compute the expansion $\mathcal{Z}^f_{\leq 1}$ up to degree 1 of $\mathcal{Z}^f$ for  and to find



where the endpoints of the tangle representative lie on $\mathbb{R} \times \{0,1\}$. See [Les20, Lemma 12.34].

More precisely, Z maps the above braid $(\sigma_1)^2$ to the exponential of an element obtained by inserting a combination $2\breve{\alpha}$ of Jacobi diagrams with two free univalent vertices, which are symmetric with respect to the exchange of these two vertices, on the diagram with one edge between the two strands. See [Les20, Lemma 12.21]. The degree one part of $2\breve{\alpha}$ is an edge between the two vertices, and it is conjectured that $2\breve{\alpha}$ vanishes in degree higher than 1. Inserting $2\breve{\alpha}$ on the edge of  gives rise to 2α, where $\alpha \in \mathcal{A}([0,1])$ is the *Bott and Taubes anomaly,* which governs the dependence on $I_\theta(K,\tau)$ as follows.

Theorem 3.4 *Let L be a long tangle representative and let L_C denote the set of its connected components. The expression*

$$\prod_{K \in L_C} (\exp(-I_\theta(K,\tau)\alpha)\sharp_K)\, Z(\check{R}, L, (I_\theta(K,\tau))_{K \in L_C})$$

is independent of the $I_\theta(K,\tau)$. It is denoted by $\mathcal{Z}(\mathcal{C}, L)$.

Here $\exp(-I_\theta(K,\tau)\alpha)$ acts on $Z(\check{R}, L, (I_\theta(K,\tau))_{K \in L_C})$, on the component of K in the source $\mathcal{L}$ of the long tangle as indicated[15] *by the subscript K. When $L = (K)_{K \in L_C}$ is framed by some $L_\parallel = (K_\parallel)_{K \in L_C}$, set*

$$\mathcal{Z}^f(\mathcal{C}, (L, L_\parallel)) = \prod_{K \in L_C} \left(\exp(lk(K, K_\parallel)\alpha)\sharp_K\right) \mathcal{Z}(\mathcal{C}, L).$$

[15] Because of the mentioned symmetry of α, there is no need to orient K to define $(\exp(-I_\theta(K,\tau)\alpha)\sharp_K)$.

Remark 3.5 It is known that $\alpha_{2n} = 0$ for any $n \in \mathbb{N}$, and that $\alpha_3 = 0$ [Poi02, Proposition 1.4]. Sylvain Poirier also found that $\alpha_5 = 0$ with the help of a Maple program. Furthermore, according to [Les02, Corollary 1.4], α_{2n+1} is a combination of diagrams with two univalent vertices (as mentioned above), and $\mathcal{Z}(S^3, L)$ is obtained from the Kontsevich integral Z^K by inserting d times the plain part $2\check{\alpha}$ of 2α on some edge of each degree d connected component of a diagram. See [Les20, Section 10.3] for more about the anomaly α, which is unknown in odd degrees greater than 6.

Let us now discuss some properties of the invariant $\mathcal{Z}^f$ of framed tangles. The first one is the following functoriality property, which is a part of [Les20, Theorem 12.18], which is proved in [Les20, Section 16.2]

Theorem 3.6 *$\mathcal{Z}^f$ is* functorial: *For two tangles $L_1 = (\mathcal{C}_1, L_1)$ and $L_2 = (\mathcal{C}_2, L_2)$ such that the top configuration of L_1 coincides with the bottom configuration of L_2, then the product L_1L_2 is defined by stacking L_2 on top of L_1, (and appropriately vertically rescaling,) and*

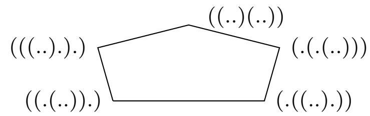
$$\mathcal{Z}^f(L_1L_2) = \mathcal{Z}^f\left(\begin{array}{|c|}\hline L_2 \\ \hline L_1 \\ \hline\end{array}\right) = \begin{array}{|c|}\hline \mathcal{Z}^f(L_2) \\ \hline \mathcal{Z}^f(L_1) \\ \hline\end{array} = \mathcal{Z}^f(L_1)\mathcal{Z}^f(L_2).$$

When applied to the case where the tangles are empty, this theorem implies that the invariant Z of $\mathbb{Q}$–spheres is multiplicative under connected sum.

3.3 Generalization to q–tangles

Here, framed tangles are cobordisms in $\mathbb{Q}$-cylinders between injective configurations of points in $\mathbb{C}$ up to dilations and translations. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and for a finite set B, the space $\check{\mathcal{S}}_B(\mathbb{K})$ of injective maps from B to $\mathbb{K}$ up to translation and dilation, may be compactified to a manifold $\mathcal{S}_B(\mathbb{K})$ by first embedding $\check{\mathcal{S}}_B(\mathbb{K})$ into the compact space of non-constant maps from B to $\mathbb{K}$ up to translation and dilation (when $\sharp B \geq 2$), and then successively blowing up all the diagonals as in the beginning of Section 2.4. See [Les20, Section 8.2] for details.

Example 3.7 For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the configuration space $\check{\mathcal{S}}_{\underline{1}}(\mathbb{K}) = \mathcal{S}_{\underline{1}}(\mathbb{K})$ is reduced to a point. The configuration space $\check{\mathcal{S}}_{\underline{2}}(\mathbb{C}) = \mathcal{S}_{\underline{2}}(\mathbb{C})$ is a circle, while the configuration space $\check{\mathcal{S}}_{\underline{2}}(\mathbb{R}) = \mathcal{S}_{\underline{2}}(\mathbb{R})$ has two points $(0, 1)$ and $(0, -1)$, where we write elements of $\check{\mathcal{S}}_{\underline{k}}(\mathbb{R})$ as elements $(c(1), \dots, c(k))$ of $\mathbb{R}^k$ such that $c(1) = 0$ and $|c(k)| = 1$, for any $k \in \mathbb{N}$ such that $k \geq 2$. In general, $\check{\mathcal{S}}_{\underline{k}}(\mathbb{R})$ and its compactification $\mathcal{S}_{\underline{k}}(\mathbb{R})$ have $k!$ components, which correspond to the orders of the $c(i)$ in $\mathbb{R}$. Denote the connected component of $\check{\mathcal{S}}_{\underline{k}}(\mathbb{R})$ where $c(1) < c(2) < \ldots < c(k)$ by $\check{\mathcal{S}}_{<,\underline{k}}(\mathbb{R})$, and its closure in $\mathcal{S}_{\underline{k}}(\mathbb{R})$ by $\mathcal{S}_{<,\underline{k}}(\mathbb{R})$. Then $\check{\mathcal{S}}_{<,\underline{3}}(\mathbb{R}) = \{(0, t, 1) \mid t \in]0, 1[\}$, and $\mathcal{S}_{<,\underline{3}}(\mathbb{R})$ is its natural compactification $[0, 1]$ where $t \in]0, 1[$ represents the injective configuration $(0, t, 1)$, 0 represents the limit configuration $((..).) = \lim_{t\to 0}(0, t, 1)$ and 1 represents the limit configuration $(.(..)) = \lim_{t\to 0}(0, 1 - t, 1)$. The configuration space $\mathcal{S}_{<,\underline{4}}(\mathbb{R})$ is diffeomorphic to the following well-known pentagon.



In general, for $k \geq 3$, the configuration space $\mathcal{S}_{<,\underline{k}}(\mathbb{R})$ is a *Stasheff polyhedron* of dimension $(k-2)$ whose corners are labeled by *non-associative words* in the letter . as in the example above. For any integer $k \geq 2$, a non-associative word w with k letters represents a limit configuration $w = \lim_{t\to 0} w(t)$, where $w(t) = (w_1(t) = 0, w_2(t), \dots, w_{k-1}(t), w_k(t) = 1)$ is an injective configuration for $t \in]0,1[$, and, if w is the product uv of a non-associative word u of length $j \geq 1$ and a non-associative word v of length $(k-j) \geq 1$, $w_i(t) = tu_i(t)$ when $1 < i \leq j$ and $w_i(t) = 1 - t + tv_{i-j}(t)$ when $k > i > j$. For example, $(((..).).)(t) = (0, t^2, t, 1)$. In a limit configuration associated to such a non-associative word, points inside matching parentheses are thought of as infinitely closer to each other than they are to points outside these matching parentheses.

Definition 3.8 Define a *combinatorial q–tangle* as a framed tangle representative whose bottom and top configurations are on the real line, up to isotopies of $\mathcal{C}$ which globally preserve the intersection of the bottom disk $D^2 \times \{0\}$ with $\mathbb{R} \times \{0\}$ and the intersection of the top disk $D^2 \times \{1\}$ with $\mathbb{R} \times \{1\}$, equipped with non-associative words of the appropriate length associated to the bottom and top configurations. These non-associative words are called the bottom and top configurations of the combinatorial q–tangle.

Such a combinatorial q–tangle L from a bottom word w^- to a top word w^+ is thought of as the limit when t approaches 0 of the framed tangles $L(t)$ in the above isotopy class whose bottom and top configurations are $w^-(t)$ and $w^+(t)$, respectively.

In [Les20, Theorem 12.39 and Remark 12.42], following Poirier [Poi00], I proved that $\lim_{t\to 0} \mathcal{Z}^f(L(t))$ exists and that it defines an isotopy invariant of these (framed) combinatorial q–tangles. The obtained invariant is still multiplicative under vertical composition as in Theorem 3.6, and we can now define other interesting operations.

For two combinatorial q–tangles $L_1 = (\mathcal{C}_1, L_1)$ from w_1^- to w_1^+ and $L_2 = (\mathcal{C}_2, L_2)$ from w_2^- to w_2^+ define the product $L_1 \otimes L_2$ from the bottom configuration $w_1^- w_2^-$ to the top configuration $w_1^+ w_2^+$ by shrinking $\mathcal{C}_1$ and $\mathcal{C}_2$ to make them respectively replace the products by $[0,1]$ of the horizontal disks with radius $\frac{1}{4}$ and respective centers $-\frac{1}{2}$ and $\frac{1}{2}$.

Theorem 3.9 *$\mathcal{Z}^f$ is* monoidal: *For two combinatorial q–tangles L_1 and L_2,*

$$\mathcal{Z}^f(L_1 \otimes L_2) = \mathcal{Z}^f\left(\boxed{L_1 \mid L_2}\right) = \boxed{\mathcal{Z}^f(L_1) \mid \mathcal{Z}^f(L_2)} = \mathcal{Z}^f(L_1) \otimes \mathcal{Z}^f(L_2).$$

Proof: This theorem can be easily deduced from the cabling property and the functoriality property of [Les20, Theorem 12.18]. □

We can also *double* a component K according to its parallelization in a combinatorial q–tangle L. This operation replaces a component with two parallel components, and, if this component has boundary points, it replaces the corresponding letters in the non-associative words with (..). The obtained combinatorial q–tangle is denoted by $L(2 \times K)$.

The corresponding operation for Jabobi diagrams is the following one.

Definition 3.10 Let $\mathcal{L}$ be a one-manifold, and let $\mathcal{K}$ be a connected component of $\mathcal{L}$. Let

$$\mathcal{L}(2 \times \mathcal{K}) = (\mathcal{L} \setminus \mathcal{K}) \sqcup \left(\mathcal{K}^{(1)} \sqcup \mathcal{K}^{(2)}\right)$$

be the manifold obtained from $\mathcal{L}$ by *duplicating* $\mathcal{K}$, that is by replacing $\mathcal{K}$ with two copies $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ of $\mathcal{K}$ and let

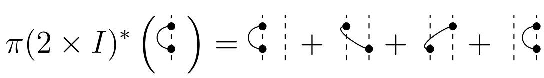
$$\pi(2 \times \mathcal{K})\colon \mathcal{L}(2 \times \mathcal{K}) \longrightarrow \mathcal{L}$$

be the associated trivial covering, which is the identity on $(\mathcal{L} \setminus \mathcal{K})$, and the trivial 2-fold covering from $\mathcal{K}^{(1)} \sqcup \mathcal{K}^{(2)}$ to $\mathcal{K}$.

If Γ is (the class of) an oriented Jacobi diagram on $\mathcal{L}$, then $\pi(2 \times \mathcal{K})^*(\Gamma)$ is the sum of all diagrams on $\mathcal{L}(2 \times \mathcal{K})$ obtained from Γ by lifting each univalent vertex to one of its preimages under $\pi(2 \times \mathcal{K})$. (These diagrams have the same vertices and edges as Γ and the local orientations at univalent vertices are naturally induced by the local orientations of the corresponding univalent vertices of Γ.) This operation induces the natural linear *duplication map:*

$$\pi(2 \times \mathcal{K})^* : \mathcal{A}(\mathcal{L}) \longrightarrow \mathcal{A}(\mathcal{L}(2 \times \mathcal{K})).$$

Example 3.11



We can now state the following duplication property for $\mathcal{Z}^f$ of [Les20, Theorem 12.18], which is proved in [Les20, Section 16.4].

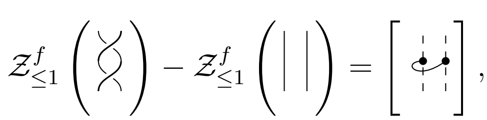
Theorem 3.12 *Let K be a component of a combinatorial q–tangle L, then*

$$\mathcal{Z}^f(L(2 \times K)) = \pi(2 \times K)^* \mathcal{Z}^f(L)$$

More properties of $\mathcal{Z}^f$ are presented in [Les20, Theorem 12.18].

3.4 Discrete derivatives of $\mathcal{Z}^f$

Since



where the endpoints of the tangles lie on $\mathbb{R} \times \{0, 1\}$, the properties above of $\mathcal{Z}^f$ allow us to completely compute n^{th} derivatives of $\mathcal{Z}_n$, where a simple derivative of $\mathcal{Z}_n$ is a difference $\mathcal{Z}_n(\times) - \mathcal{Z}(\times)$. In particular, they imply that the restriction of $\mathcal{Z}$ to links in S^3 is a universal

Vassiliev invariants of links as in [Les20, Section 16.6], without using the theorem mentioned in Remark 3.5.

The following n^{th} derivative with respect to LP-surgeries of $\mathcal{Z}^f$ is computed in [Les20, Theorem 17.5]. Let L be a q–tangle representative in a rational homology cylinder $\mathcal{C}$. Let $\sqcup_{i=1}^{x} A^{(i)}$ be a disjoint union of rational homology handlebodies embedded in $\mathcal{C} \setminus L$. Let $(A^{(i)\prime}/A^{(i)})$ be rational LP surgeries in $\mathcal{C}$ as in Definition 1.26. Set $X = [\mathcal{C}, L; (A^{(i)\prime}/A^{(i)})_{i \in \underline{x}}]$ and

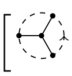
$$\mathcal{Z}_n(X) = \sum_{I \subset \underline{x}} (-1)^{x + \sharp I} \mathcal{Z}_n\left(\mathcal{C}_I, L\right),$$

where $\mathcal{C}_I = \mathcal{C}\left((A^{(i)\prime}/A^{(i)})_{i \in I}\right)$ is the rational homology cylinder obtained from $\mathcal{C}$ by performing the LP-surgeries that replace $A^{(i)}$ with $A^{(i)\prime}$ for $i \in I$. If $2n < x$, then $\mathcal{Z}_n(X)$ vanishes, and, if $2n = x$, then the expression of $\mathcal{Z}_n(X)$ is given in [Les20, Theorem 17.5].

This computation relies on constructions of propagating forms that coincide as much as possible for the involved manifolds. The result of this computation implies that the restriction of Z to $\mathbb{Q}$-spheres is a universal finite invariant of $\mathbb{Q}$-spheres with respect to the Moussard theory of finite invariants of $\mathbb{Q}$-spheres [Mou12], as announced in Section 3.1.

This computation also allowed the author to compute $\check{Z}_2(R, K)$ for any null-homologous knot K in a rational homology sphere R in [Les20, Theorem 17.36], and to find

$$\check{Z}_2(R, K) = \left(\frac{1}{24} - \frac{1}{2}\Delta_K''(1)\right)\left[\quad\right],$$



where Δ_K is the Alexander polynomial of K, normalized so that $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$.

3.5 Some open questions

The determinations of the anomalies α and β are still open. The behaviour of $\mathcal{Z}^f$ under Dehn surgeries has not yet been investigated. Is the invariant $\mathcal{Z}$ of $\mathbb{Q}$-spheres obtained from the invariant $\mathcal{Z}^f$ of framed links in the same way as the Le-Murakami-Ohtsuki invariant [LMO98] is obtained from the Kontsevich integral ?

I constructed an invariant $\tilde{\mathcal{Z}}$ of null-homologous knots in $\mathbb{Q}$-spheres from equivariant algebraic intersections in equivariant configuration spaces in [Les11, Les13]. This equivariant $\tilde{\mathcal{Z}}$ lives in a more structured space of Jacobi diagrams. It shares many properties with the Kricker lift of the Kontsevich integral of [Kri00, GK04]. Does $\tilde{\mathcal{Z}}$ lift the restriction of $\mathcal{Z}$ to null-homologous knots in $\mathbb{Q}$-spheres as the Kricker invariant lifts the Kontsevich integral ?

Heegaard splittings provide propagators as in Section 1.5. How do the invariants Z, $\mathcal{Z}^f$ and $\tilde{\mathcal{Z}}$ relate to Heegaard-Floer homology ?

References

[AM90] S. Akbulut & J. D. McCarthy – *Casson's invariant for oriented homology* 3-*spheres*, Mathematical Notes, vol. 36, Princeton University Press, Princeton, NJ, 1990, An exposition.

[AS94] S. Axelrod & I. M. Singer – "Chern-Simons perturbation theory. II", *J. Differential Geom.* **39** (1994), no. 1, p. 173–213.

[BCR98] J. Bochnak, M. Coste & M.-F. Roy – *Real algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 36, Springer-Verlag, Berlin, 1998, Translated from the 1987 French original, Revised by the authors.

[BN95a] D. Bar-Natan – "On the Vassiliev knot invariants", *Topology* **34** (1995), no. 2, p. 423–472.

[BN95b] — , "Perturbative Chern-Simons theory", *J. Knot Theory Ramifications* **4** (1995), no. 4, p. 503–547.

[BT94] R. Bott & C. Taubes – "On the self-linking of knots", *J. Math. Phys.* **35** (1994), no. 10, p. 5247–5287, Topology and physics.

[CDM12] S. Chmutov, S. Duzhin & J. Mostovoy – *Introduction to Vassiliev knot invariants*, Cambridge University Press, Cambridge, 2012.

[Fuk96] K. Fukaya – "Morse homotopy and Chern-Simons perturbation theory", *Comm. Math. Phys.* **181** (1996), no. 1, p. 37–90.

[Gau77] C. F. Gauss – "Zur mathematischen Theorie der electrodynamischen Wirkungen, manuscript, first published in his Werke Vol. 5", *Königl. Ges. Wiss. Göttingen, Göttingen* (1877), p. 601–630.

[GGP01] S. Garoufalidis, M. Goussarov & M. Polyak – "Calculus of clovers and finite type invariants of 3-manifolds", *Geom. Topol.* **5** (2001), p. 75–108.

[GK04] S. Garoufalidis & A. Kricker – "A rational noncommutative invariant of boundary links", *Geom. Topol.* **8** (2004), p. 115–204.

[GM92] L. Guillou & A. Marin – "Notes sur l'invariant de Casson des sphères d'homologie de dimension trois", *Enseign. Math. (2)* **38** (1992), no. 3-4, p. 233–290, With an appendix by Christine Lescop.

[GMM90] E. Guadagnini, M. Martellini & M. Mintchev – "Wilson lines in Chern-Simons theory and link invariants", *Nuclear Phys. B* **330** (1990), no. 2-3, p. 575–607.

[Hir73] F. E. P. Hirzebruch – "Hilbert modular surfaces", *Enseignement Math. (2)* **19** (1973), p. 183–281.

[Hir94] M. W. Hirsch – *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1994, Corrected reprint of the 1976 original.

[KM99] R. Kirby & P. Melvin – "Canonical framings for 3-manifolds", in *Proceedings of 6th Gökova Geometry-Topology Conference*, vol. 23, Turkish J. Math., no. 1, 1999, p. 89–115.

[Kon94] M. Kontsevich – "Feynman diagrams and low-dimensional topology", in *First European Congress of Mathematics, Vol. II (Paris, 1992)*, Progr. Math., vol. 120, Birkhäuser, Basel, 1994, p. 97–121.

[Kri00] A. Kricker – "The lines of the Kontsevich integral and Rozansky's rationality conjecture", arXiv:math/0005284, 2000.

[KT99] G. Kuperberg & D. Thurston – "Perturbative 3–manifold invariants by cut-and-paste topology", math.GT/9912167, 1999.

[Kui99] N. H. Kuiper – "A short history of triangulation and related matters", in *History of topology*, North-Holland, Amsterdam, 1999, p. 491–502.

[Les02] C. Lescop – "About the uniqueness of the Kontsevich integral", *J. Knot Theory Ramifications* **11** (2002), no. 5, p. 759–780.

[Les04] — , "Splitting formulae for the Kontsevich-Kuperberg-Thurston invariant of rational homology 3-spheres", math.GT/0411431, 2004.

[Les05] C. Lescop – "Knot invariants and configuration space integrals", in *Geometric and topological methods for quantum field theory*, Lecture Notes in Phys., vol. 668, Springer, Berlin, 2005, p. 1–57.

[Les11] C. Lescop – "Invariants of knots and 3-manifolds derived from the equivariant linking pairing", in *Chern-Simons gauge theory: 20 years after*, AMS/IP Stud. Adv. Math., vol. 50, Amer. Math. Soc., Providence, RI, 2011, p. 217–242.

[Les13] — , "A universal equivariant finite type knot invariant defined from configuration space integrals", arXiv:1306.1705, 2013.

[Les15a] — , "A formula for the Θ-invariant from Heegaard diagrams", *Geom. Topol.* **19** (2015), no. 3, p. 1205–1248.

[Les15b] — , "On homotopy invariants of combings of three-manifolds", *Canad. J. Math.* **67** (2015), no. 1, p. 152–183.

[Les20] — , "Invariants of links and 3–manifolds from graph configurations", arXiv:2001.09929, 2020.

[LMO98] T. T. Q. Le, J. Murakami & T. Ohtsuki – "On a universal perturbative invariant of 3-manifolds", *Topology* **37** (1998), no. 3, p. 539–574.

[Mar88] A. Marin – "Un nouvel invariant pour les sphères d'homologie de dimension trois (d'après Casson)", *Astérisque* (1988), no. 161-162, p. 151–164, Exp. No. 693, 4 (1989), Séminaire Bourbaki, Vol. 1987/88.

[Mil97] J. W. Milnor – *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Based on notes by David W. Weaver, Revised reprint of the 1965 original.

[Mou12] D. Moussard – "Finite type invariants of rational homology 3-spheres", *Algebr. Geom. Topol.* **12** (2012), p. 2389–2428.

[Oht96] T. Ohtsuki – "Finite type invariants of integral homology 3-spheres", *J. Knot Theory Ramifications* **5** (1996), no. 1, p. 101–115.

[Poi00] S. Poirier – "The configuration space integral for links and tangles in $\mathbb{R}^3$", arXiv:math/0005085v2, 2000.

[Poi02] S. Poirier – "The configuration space integral for links in $\mathbb{R}^3$", *Algebr. Geom. Topol.* **2** (2002), p. 1001–1050.

[Pol05] M. Polyak – "Feynman diagrams for pedestrians and mathematicians", in *Graphs and patterns in mathematics and theoretical physics*, Proc. Sympos. Pure Math., vol. 73, Amer. Math. Soc., Providence, RI, 2005, p. 15–42.

[Saw06] J. Sawon – "Perturbative expansion of Chern-Simons theory", in *The interaction of finite-type and Gromov-Witten invariants (BIRS 2003)*, Geom. Topol. Monogr., vol. 8, Geom. Topol. Publ., Coventry, 2006, p. 145–166.

[Thu99] D. Thurston – "Integral expressions for the Vassiliev knot invariants", math.QA/9901110, 1999.

[Vog11] P. Vogel – "Algebraic structures on modules of diagrams", *J. Pure Appl. Algebra* **215** (2011), no. 6, p. 1292–1339.

[Wal92] K. Walker – *An extension of Casson's invariant*, Annals of Mathematics Studies, vol. 126, Princeton University Press, Princeton, NJ, 1992.

[Wat18] T. Watanabe – "Higher order generalization of Fukaya's Morse homotopy invariant of 3-manifolds I. Invariants of homology 3-spheres", *Asian J. Math.* **22** (2018), no. 1, p. 111–180.

[Wit89] E. Witten – "Quantum field theory and the Jones polynomial", *Comm. Math. Phys.* **121** (1989), no. 3, p. 351–399.