The Hurwitz existence problem
for surface branched covers

WinterBraids X – Pisa – Feb. 17-20, 2020

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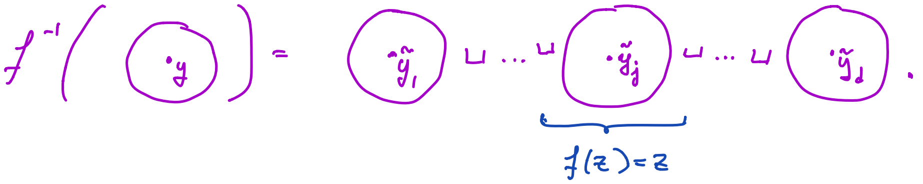
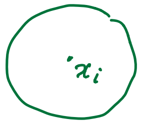
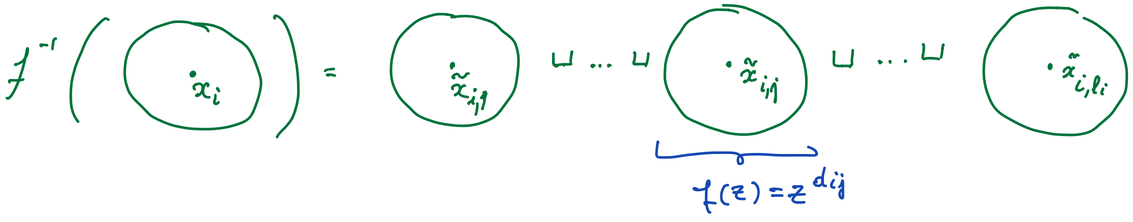
Closed surfaces: $S, T, \mathbb{P}, g\cdot T, k\cdot\mathbb{P}$.

Branched cover : $f:\tilde{\Sigma}\to\Sigma$ continuous s.t.
$\exists\,\{x_i\}_{i=1}^{n}\subset\Sigma$ with $f^{-1}(x_i)$ finite and
$\underbrace{f|_{\dots}:\tilde{\Sigma}\setminus\{f^{-1}(x_i)\}\to\Sigma\setminus\{x_i\}}_{f^{\bullet}:\tilde{\Sigma}^{\bullet}\to\Sigma^{\bullet}}$ ordinary degree-d cover

i.e. $\forall y\neq x_i$



Now  $\setminus\, x_i \simeq S^1$ whence



$\pi_i=[d_{ij}]_{j=1}^{\ell_i}$ partition of d

$\pi=[\pi_i]_{i=1}^{n}$ $\quad\tilde{n}=\sum_{i=1}^{n}\ell_i$

$\pi_i\neq[1,\dots,1]$

Branch datum associated to f : $(\tilde{\Sigma}, \Sigma, d, n; \pi)$

Necessary conditions:

- $\chi(\tilde{\Sigma}) - \tilde{n} = d \cdot (\chi(\Sigma) - n)$ Riemann-Hurwitz
- $n \cdot d \equiv \tilde{n} \pmod 2$
- Σ orientable $\Longrightarrow$ $\tilde{\Sigma}$ too
- Σ non-orientable, $\tilde{\Sigma}$ orientable

  $\Longrightarrow$ d even, $\pi_i = [\pi_i', \pi_i'']$ partitions of $d/2$

Because if $\bar{\Sigma} \xrightarrow{2:1} \Sigma$ is orientation cover

$$\begin{tikzcd}
\tilde{\Sigma}^\bullet \arrow[r, "g^\bullet"] \arrow[dr, "f^\bullet"'] & \bar{\Sigma}^\bullet \arrow[d] \\
 & \Sigma^\bullet
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
\tilde{\Sigma} \arrow[r, "g"] \arrow[dr, "f"'] & \bar{\Sigma} \arrow[d] \\
 & \Sigma
\end{tikzcd}$$

$f^{-1}(x_i) = g^{-1}(x_i') \sqcup g^{-1}(x_i'')$.

$n \cdot d \equiv \tilde{n} \pmod 2$ obvious unless $\tilde{\Sigma}, \Sigma$ non-orientable.

Hurwitz problem : given $(\tilde{\Sigma}, \Sigma, d, n; \pi)$,
is it associated to an existing f ?

Examples:

- $(\mathbb{P}, \mathbb{P}, 6, 2; [2,2,1,1], [3,2,1])$ RH: $1-7 = 6(1-2)$

  $2\cdot 6 \not\equiv 7 \pmod 2$

- $(2\mathbb{P}, \mathbb{P}, 7, 3; [3211], [31111], [22111])$ RH: $0-14 = 7(1-3)$

  $3\cdot 7 \not\equiv 14 \pmod 2$

- $(S, \mathbb{P}, 6, 2; [4,1,1], [21111])$ RH: $2-8 = 6(1-2)$

  $[4,1,1] \neq (\underbrace{[\ldots]}_{3}, \underbrace{[\ldots]}_{3})$

Setting: for orientable $\tilde{\Sigma}, \Sigma$ can ask same question for

- $\tilde{\Sigma}, \Sigma$ complex curves, f holomorphic
- $\tilde{\Sigma}, \Sigma$ projective 1-varieties, f rational

Same answer!

Counting: $\#\{f : \text{realizing } (\tilde{\Sigma}, \Sigma, d, n; \pi)\}/_{\sim}$

$f_1 \sim f_2$ if $\exists$
$$\begin{tikzcd}
\tilde{\Sigma} \arrow[r, "\tilde{h}"] \arrow[d, "f_1"'] & \tilde{\Sigma} \arrow[d, "f_2"] \\
\Sigma \arrow[r, "h"] & \Sigma
\end{tikzcd}$$

- $h = id$ strong ν_s
- $h, \tilde{h}$ positive weak ν_w
- $\forall h, \tilde{h}$ very weak ν_v

▷ ν_s computed implicitly by Mednykh

▷ $\nu_s = \nu_w$ if π has no repetitions

▷ $\nu_s \geq \nu_w \geq \nu_v$ and all $\nu \left(\overset{=}{>}\right) \nu_w \left(\overset{=}{>}\right) \nu_v$ occur.

[PS19]

Problem: $(\tilde{\Sigma}, \Sigma, d, n; \pi)$ realizable or exceptional?

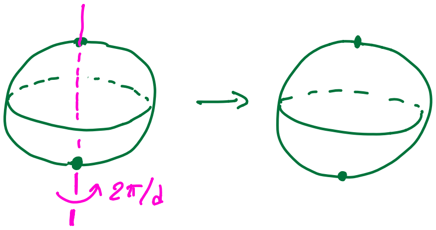
$\tilde{g}$ $\quad$ g

Orientable cases with small n:

$g = 0$ $\quad n \leq 1$ ... easy

$g = 0$ $\quad n = 2$ $\quad (S, S, d, 2; [d], [d])$

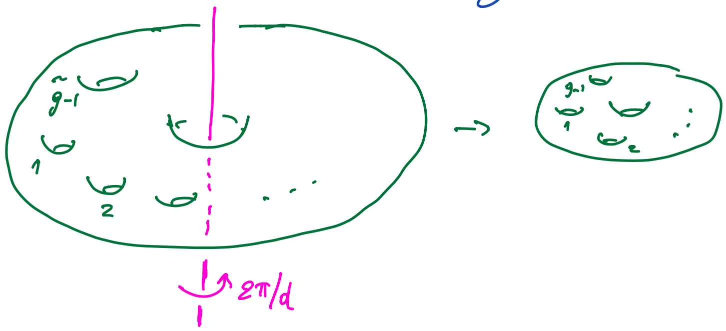
realizable



$g = 0$ $\quad n \geq 3$ ... difficult

$g > 0$, $n = 0$ $\quad 2(1-\tilde{g}) = d \cdot 2(1-g)$ $\quad \tilde{g} = 1 + d \cdot (g-1)$

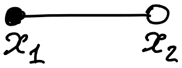
realizable



$g > 0$, $n \geq 1$ ... difficult

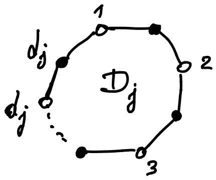
Exceptional data from dessins d'enfant

$(\tilde{\Sigma}, S, d, 3; \pi)$ realized by f

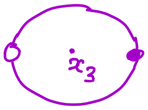
consider 

and $\Gamma = f^{-1}($•——○$)$;

graph on $\tilde{\Sigma}$ s.t.

- ▹ Γ bipartite 
- ▹ val(• … •) $= \pi_1$ val(○ … ○) $= \pi_2$
- ▹ $\tilde{\Sigma} \setminus \Gamma$ = 

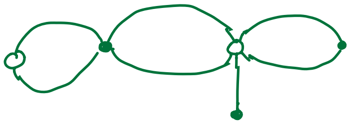
$(d_j) = \pi_3$

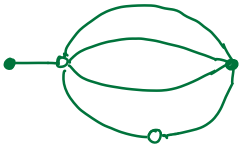
because $S \setminus$ •——○ = 

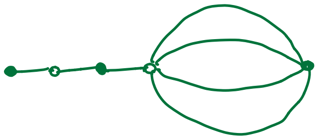
Conversely, if Γ exists we can define f (first on Γ and then extend to discs).

Example: $(\tilde{\Sigma}, S, 7, 3; [4,2,1], [5,2], \pi_3)$

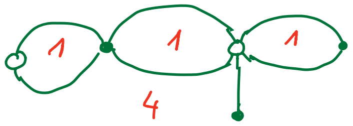
Abstract Γ :

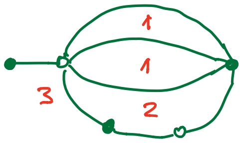


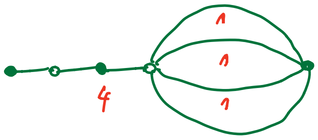


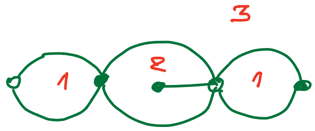


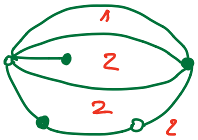
Embeddings in $\mathbb{S}$:

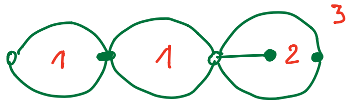


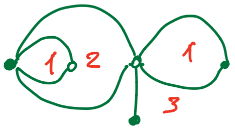


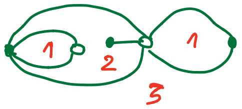


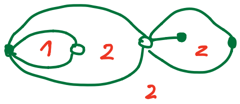




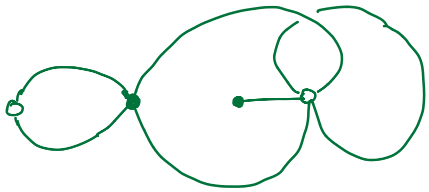


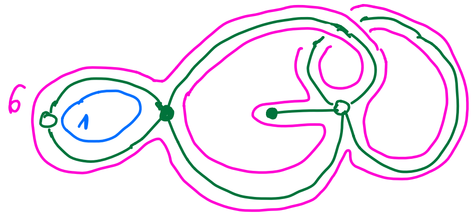


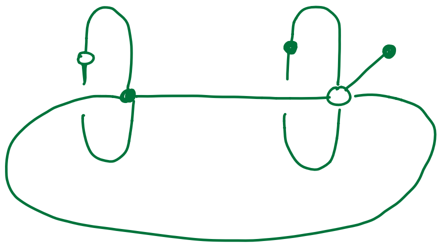


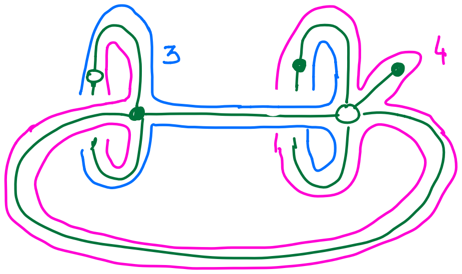


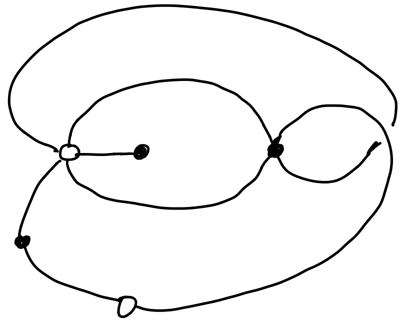
Some embeddings in T :

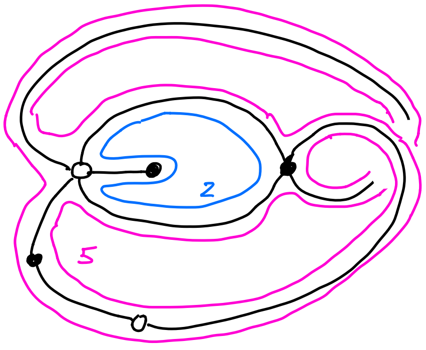






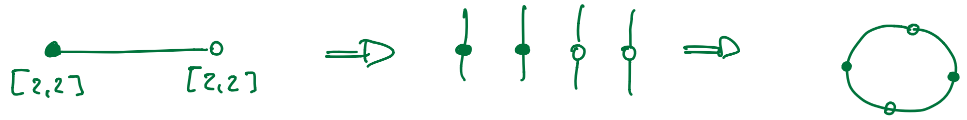




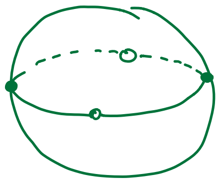


⟹ all realizable.

Example : $(S, S, 4, 3; [2,2], [2,2], [3,1])$ exceptional



⟹



⟹ $\pi_3 = [2,2]$.

The monodromy approach

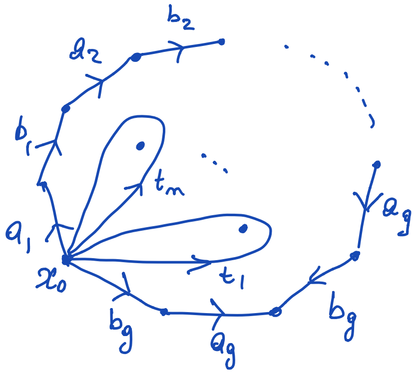
[Hur1891]

Thm : $(\tilde{\Sigma}, gT, d, m; \pi)$ realizable iff
$\exists\ \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \theta_1, \dots, \theta_m \in \mathfrak{S}_d$ s.t.

- $\prod_{p=1}^{g} [\alpha_p, \beta_p] \cdot \prod_{i=1}^{m} \theta_i = \mathrm{id}$
- the cycles of θ_i have lengths π_i
- $\langle \alpha_p, \beta_p, \theta_i \rangle$ acts transitively on $\{1, \dots, d\}$

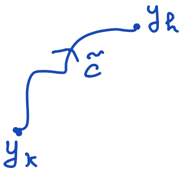
Proof: Given f, take $f^\circ : \tilde{\Sigma}^\circ \to (gT)^\circ$

Note $\pi_1((gT)^\circ, x_0) = \langle a_1, b_1, \dots, a_g, b_g, t_1, \dots, t_m : \prod [a_p, b_p] \cdot \prod t_i \rangle$

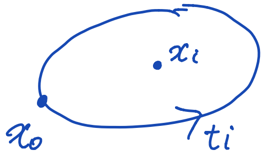


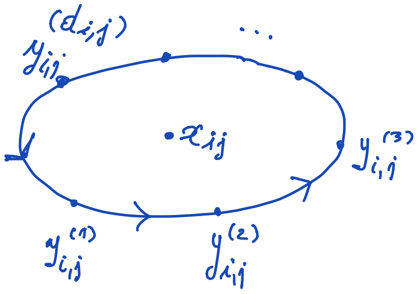
$f^{-1}(x_0) = \{y_1, \dots, y_d\}$

$\rho : \pi_1((gT)^\circ, x_0) \longrightarrow \mathfrak{S}_d$ $\quad \rho(c)(k) = h$ if

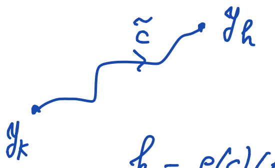


$\rho(a_p) = \alpha_p \qquad \rho(b_p) = \beta_p \qquad \rho(t_i) = \theta_i$

- $\prod [\alpha_p, \beta_p] \cdot \prod \theta_i = id$
- 

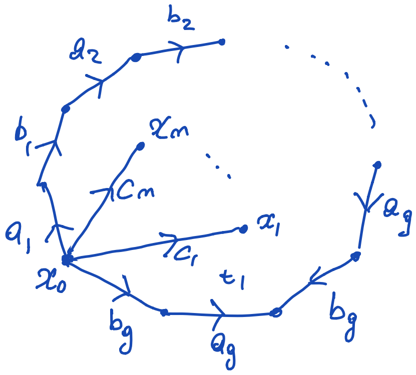
$\rightsquigarrow$


- $\langle \alpha_p, \beta_p, t_i \rangle$ transitive because

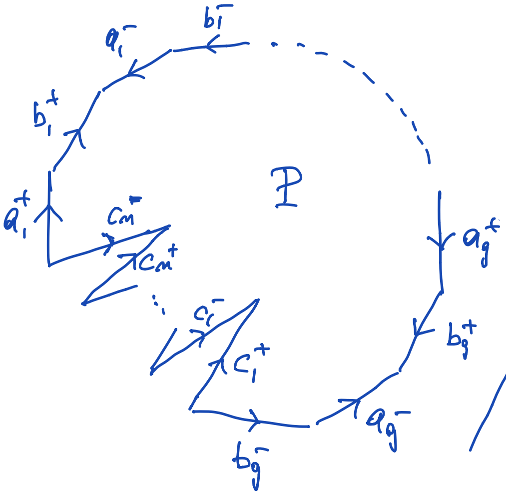


$h = \rho(c)(k).$

Conversely



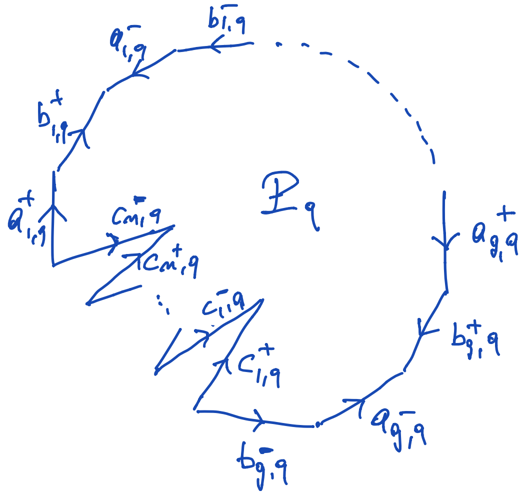
$\Longrightarrow \Sigma =$



$/$

$a_p^+ = a_p^-$
$b_p^+ = b_p^-$
$c_i^+ = c_i^-$

$\tilde{\tilde{\Sigma}} = \bigsqcup_{q=1}^{g}$



$/$

$$a_{p,q}^+ = a^-_{p,\alpha_p(q)}$$
$$b_{p,q}^+ = b^-_{p,\beta_p(q)}$$
$$c_{i,q}^+ = c^-_{i,\theta_i(q)}$$

$f : \tilde{\tilde{\Sigma}} \to \Sigma$ induced by $P_q \xrightarrow{id} P$ $\forall q$

$\tilde{\tilde{\Sigma}}$ connected by transitivity

f branched cover realizing $(\tilde{\tilde{\Sigma}}, \Sigma, d, m; \pi)$

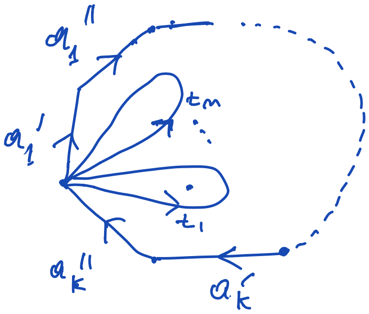
$\Longrightarrow \chi(\tilde{\tilde{\Sigma}}) = \chi(\tilde{\Sigma})$ because determined by Σ, d, m, π

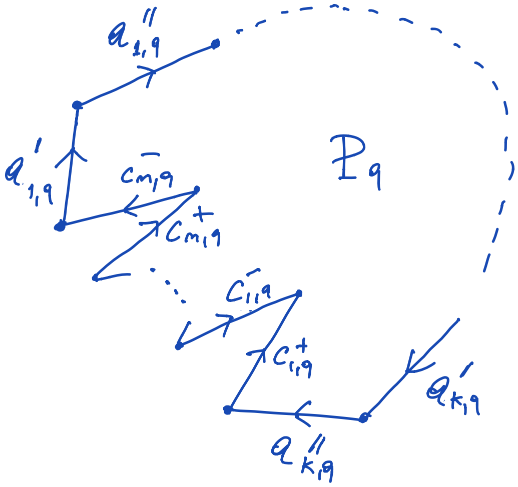
both orientable $\Longrightarrow \tilde{\tilde{\Sigma}} = \tilde{\Sigma}$. ▨

Thm: $(\tilde{\Sigma}, k\mathbb{P}, d, m; \pi)$ realizable iff $\exists\ \alpha_1, \dots, \alpha_k, \theta_1, \dots, \theta_m \in \mathfrak{S}_d$ s.t.

- $\prod \alpha_p^2 \cdot \prod \theta_i = id$
- the cycles of θ_i have lengths π_i
- $\langle \alpha_p, \theta_i \rangle$ transitive on $\{1, \dots, d\}$
- each α_p contains cycles of even length only $\Longleftrightarrow$ $\tilde{\Sigma}$ orientable

Proof: similar with

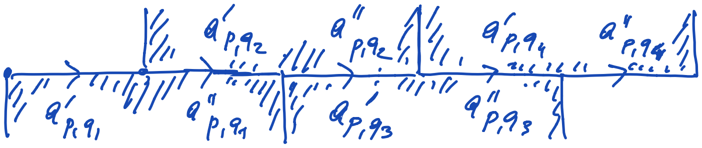




$a'_{p,q} = a''_{p,\alpha_p(q)}$

$c^+_{i,q} = c^-_{i,\theta_i(q)}$

$\tilde{\Sigma}$ orientable $\Longleftrightarrow$ all loops in $\tilde{\Sigma}$ arising as liftings of some α_p^m are orientation-preserving; one such loop corresponds to a cycle of length m of α_p, and it preserves orientation precisely for even m:



… $(q_1, q_2, \dots)$ cycle of d_p^{-1}

closes up to annulus for even m and to Möbius for odd m. ▨

Cor: $n \cdot d \equiv \tilde{n}$ mod 2 for a realizable datum.

Proof: $\sigma((1,\dots,m)) \equiv m-1$

$\Rightarrow \sigma(\theta_i) \equiv d - l_i$

$\Rightarrow \sigma(\theta_1 \cdots \theta_m) \equiv m \cdot d - \tilde{m}$

but $\theta_1 \cdots \theta_m$ is a product of squares. ▨

Exceptional data from monodromy

$$\big((m-3) \cdot T, S, 4, n; [2,2], \dots, [2,2], [3,1]\big)$$

$$2(1-(m-3)) - 2n = 4(2-n)$$ ✓

if $\pi(\eta) = \pi(\tau) = [2,2]$ for η, τ we have two cases:

$((12)(34)) \cdot ((12)(34)) = id$

$((12)(34)) \cdot ((13)(24)) = (14)(23)$

so by repeated multiplication I never get $[3,1]$.

[EKS84] [Ez78]
[Hus62] ...

Thm 1: $(\tilde{\Sigma}, \Sigma, d, m; \pi)$ realizable
if $\chi(\Sigma) \leq 0$ and $\tilde{\Sigma}\ \Sigma$ have the same orientability

Cor 2: $(\tilde{\Sigma}, \Sigma, d, m; \pi)$ realizable
if $\chi(\Sigma) \leq 0$ and $\tilde{\Sigma}, \Sigma$ have opposite orientability.

Proof: $\tilde{\Sigma}$ orientable, Σ non orientable; $\bar{\Sigma} \xrightarrow{2:1} \Sigma$
d even, $\pi_i = [\pi_i', \pi_i'']$.
$(\tilde{\Sigma}, \bar{\Sigma}, d/2, 2m, \{\pi', \pi''\})$ realizable
with partion π_i' over x_i' and π_i'' over x_i''

$\forall F \subset \tilde{\Sigma}$ finite, $y \neq z \notin F$ $\exists\, g : \bar{\Sigma} \to \bar{\Sigma}$
s.t. $g|_F = id_F$ $g(y) = z$ $g(z) = y$

$\Longrightarrow$ wlog x_i', x_i'' have same image in Σ. ▨

Thm 3: $(\tilde{\Sigma}, \mathbb{P}, d, m; \pi)$ realizable if $\tilde{\Sigma}$ non-orientable

Consequence: can reduce to $\Sigma = \mathbb{S}$.
Because for $\Sigma \neq \mathbb{S}$ realizable unless $\Sigma = \mathbb{P}$, $\tilde{\Sigma}$ orient.
Then realizable iff can split $\pi_i = [\pi_i', \pi_i'']$ s.t.
$(\tilde{\Sigma}, \mathbb{S}, d/2, m; \{\pi_i', \pi_i''\})$ realizable

Example: $(S, \mathbb{P}, 20, 2; [\underbrace{2,\dots,2}_{10}]\ [6,2,2,2,\underbrace{1,\dots,1}_{8}])$

$d - (10+4+8) = 20(1-2)$ ✓

Indeed π_1, π_2 split as $\pi_1', \pi_1'', \pi_2', \pi_2''$ but any such gives

$$(S, S, 10, 4;\ \underbrace{[2,\dots,2], [2,\dots,2], [6,\dots]}\ [\dots])$$

will show later this always exceptional

$\Rightarrow$ exceptional

Prop: $\theta \in \mathfrak{S}_d$ even can be written as

- $\theta = [\alpha, \beta]$ $\quad \alpha$ full cycle
- $\theta = \alpha_1^2 \alpha_2^2$ $\quad \alpha_1, \alpha_2$ full cycle

Lem: $\theta \in \mathfrak{S}_d$ with l cycles

$t \geq 0 \quad l + 2t \leq d$

$\Rightarrow$ can write $\theta = \sigma\tau$ $\quad \sigma$ d-cycle $\quad \tau$ $(l+2t)$-cycle.

Proof (Lem): $\theta = (1,\dots,a_1)(a_1+1,\dots,a_2)\dots(a_{l-1}+1,\dots,a_l)$ $\quad a_l = d$.

$\delta := (a_1,\dots,a_l, b_1,\dots,b_{2t})$

$b_1 < \dots < b_{2t} \notin \{a_1,\dots,a_l\}$

$\delta = \delta_0 \cdot \delta_1 \qquad \delta_0 = (a_1,\dots,a_{l-1}, d) \qquad \delta_1 = (b_1,\dots,b_{2t}, d)$

$$\theta \cdot \delta_0 = (1,\dots,a_1)(a_1+1,\dots,a_2)\cdots(a_{l-1}+1,\dots,a_l)(a_1,\dots,a_l)$$

$$= (1,\dots a_1, a_1+1,\dots, a_2, a_2+1,\dots\ ) = (1,\dots,d)$$

$$\theta \cdot \delta = \theta \cdot \delta_0 \cdot \delta_1 = (1, \dots, d)(b_1, \dots, b_{2t}, d)$$
$$= (d, b_1+1, \dots, b_2, b_3+1, \dots, b_4, b_5+1, \dots, b_{2t},$$
$$1, 2, \dots, b_1, b_2+1, \dots, b_{2t-1}, b_{2t}+1, \dots) =: \sigma$$

$\Rightarrow \quad \theta = \sigma \cdot \delta^{-1} \qquad \tau := \delta^{-1}.$ ▨

<u>Proof (Prop):</u> θ even with l cycles $\Rightarrow$ $d - l$ even
$\Rightarrow$ can choose $l + 2t = d$ $\Rightarrow$ $\theta = \sigma \cdot \tau$, d-cycles.

- τ, σ^{-1} conjugate $\Rightarrow$ $\tau = \beta \sigma^{-1} \beta^{-1}$
  $\Rightarrow \theta = [\sigma, \beta] \qquad \alpha := \sigma$

- σ, τ conjugate $\Rightarrow$ $\sigma = \alpha_1 \tau \alpha_1^{-1}$
  $\Rightarrow \theta = \alpha_1 \tau \alpha_1^{-1} \tau = \alpha_1^2 (\alpha_1^{-1} \tau)^2$
  $\alpha_2 := \alpha_1^{-1} \tau \qquad \alpha_1 \alpha_2 = \tau.$ ▨

<u>Proof</u> (Thm 1) : choose $\theta_i \in \mathfrak{S}_d$ with cycle lengths π_i,
set $\theta = \theta_1 \cdots \theta_m$; $md \equiv \tilde{m}$ (mod 2) $\Rightarrow$ θ even.

- $\Sigma = g \cdot T \quad g \geq 1$; use Prop for θ^{-1} to write
  $\theta^{-1} = [\alpha_1, \beta_1]$ α_1 full cycle ; choose $\alpha_2 = \dots = \beta_g = id.$
  $\Rightarrow \prod [\alpha_p, \beta_p] \cdot \prod \theta_i = id \qquad \alpha_1$ full cycle

- $\Sigma = k.\mathbb{P}$ $k \geq 2$; use Prop for θ^{-1} to write
  $\theta^{-1} = \alpha_1^2 \alpha_2^2$ $\alpha_1 \alpha_2$ full cycle ; choose $\alpha_3 = \dots = \alpha_k = id$
  $\Rightarrow \prod \alpha_p^2 \cdot \prod \theta_i = id$
  $\alpha_1 \cdot \alpha_2$ full cycle
  $\Rightarrow \overset{\approx}{\Sigma},$ connected and $\alpha_1 \cdot (\alpha_1 \cdot \alpha_2)^m$ has fixed point for some $m \Rightarrow a_1 \cdot (a_1 a_2)^m$ lifts to a loop but it is orientation reversing $\Rightarrow \overset{\approx}{\Sigma}$ as well. $\boxtimes$

For Thm 3, we can choose only α with

$\alpha^2 \theta_1 \cdots \theta_n = id$.

Not always possible for random $\theta_1, \dots, \theta_n$: select suitable representatives using more work on $\mathfrak{S}_d$.

$$\boxed{\text{From now on, } \Sigma = S}$$

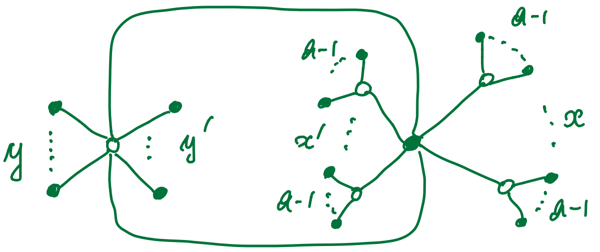
Prop: $\forall d = a \cdot b$ composite $\exists$ exceptional $(S, S, d, 3; \pi)$.

Proof: $\pi = \underbrace{[\underbrace{a,\dots,a}_{b}], [\underbrace{b+1,1,\dots,1}_{\substack{1+(ab-(b+1)) \\ = ab-b}}], [\underbrace{a, a(b-1)}_{2}]}_{ab+2}$

$[a,\dots,a]\ [b+1,1,\dots,1]$

Γ connected $\Longrightarrow$ b of the $b+1$ edges from ● go to the different ○

$\Longrightarrow \Gamma =$



$0 \le y \le a-2 \qquad 0 \le x \le b-1$

realizes : for $x=0$ $\quad 1+y \le a-1$

for $x>0$ $\quad 2+x(a-1)+y \geq 2+(a-1) \geq a+1$. ▨

Prop: given d, if all $(\tilde{\Sigma}, S, d, m; \pi)$ realizable for $m=3$, then all are _

"Proof" Induction on $m \geq 3$. For $m \geq 4$ choose θ_1, θ_2 randomly matching π_1, π_2 ; set $\theta_2' = \theta_1 \cdot \theta_2$, $\pi_2' = \pi(\theta_2')$. By induction* $\exists\, \theta_3, \dots, \theta_m$ with θ_i matching π_i, $\langle \theta_3, \dots, \theta_m \rangle$ transitive, $\theta_2' \theta_3 \cdots \theta_m = 1 \Rightarrow$ ok.

* For random θ_1, θ_2 $\nexists\, (\tilde{\Sigma}', S, d, m-1; \pi_2', \pi_3, \dots, \pi_m)$: must choose i, j, θ_i, θ_j suitably.

CONJECTURE : $(\tilde{\Sigma}, S, d, n; \pi)$ realizable for prime d. Enough to show for $m = 3$.

$\boxed{\text{From now on, } \Sigma = S \text{ and often } m = 3}$