Problem: given $(\tilde{\Sigma}, S, d, m; \pi)$

$\pi = [\pi_i]_{i=1}^{m}$ $\quad$ $\pi_i = [d_{ij}]_{j=1}^{\ell_i}$ $\quad$ $\tilde{n} = \ell_1 + \dots + \ell_m$

$(\pi_i \neq [1, \dots, 1])$

$$2(1-\tilde{g}) - \tilde{n} = d(2-m)$$

does there exist $f: \tilde{\Sigma} \to S$ matching the datum?

Results with one short partition

- Yes if $\pi_1 = [d]$ $\quad$ [Thom65], [EKS84] [KhoZnd96]

  $\downarrow$ Uniqueness via action of braid group on monodromy

- Complete classification if $\pi_1 = [k, d-k]$ for:
  - $k=1$ $\forall m$ $\forall \tilde{\Sigma}$ $\quad$ [EKS84]
  - $m=3$ $\pi_2 = \pi_3 = [2, \dots, 2]$ $(\Rightarrow \tilde{\Sigma} = S)$ $\quad$ [EKS84]
  - $m=3$, $k=2$ $\forall \tilde{\Sigma}$ $\quad$ [PePe06]
  - $\forall m$ $\forall k$ $\tilde{\Sigma} = S$ $\quad$ [Pak09]

Rem: $f(z) = a_0 + a_1 z + \cdots + a_d z^d \in \mathbb{C}[z]$ $\qquad f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$f^{-1}(\infty) = \infty \qquad \dfrac{1}{f(\frac{1}{w})} = w^d (a_d + O(w))$

$\Rightarrow f$ realizes $[d], \pi_2, \ldots, \pi_m$

Any $(S, S, d, m, \{[d], \pi_2, \ldots, \pi_m\})$ realized by $\mathbb{C}[z]$

Rem: $f(z) = a_{-h} z^{-h} + \cdots + a_0 + \cdots + a_k z^k \in \mathbb{C}[z^{\pm 1}]$

$f^{-1}(\infty) = \{0, \infty\}$ $\qquad f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$\dfrac{1}{f(\frac{1}{w})} = w^k (a_k + O(w)) \qquad \dfrac{1}{f(w)} = w^h (a_{-h} + O(w))$

$\Rightarrow f$ realizes $[k, h], \pi_2, \ldots, \pi_m$

Any realizable $(S, S, d, m; [k, d-k], \pi_2, \ldots, \pi_m)$ realized by $\mathbb{C}[z^{\pm 1}]$

Survey without proofs (large n, easy π_1)

- $m \cdot d - \tilde{m} \geq 3(d-1) \Rightarrow$ realizable except [EKS84]

$$((m-3) \cdot T, S, 4, m; [2,2], \ldots, [2,2], [3,1])$$

- $\tilde{\Sigma} = S$, $\quad \exists r$ s.t. $l_1 + \cdots + l_r = (r-1)d + 1$ [Bar01]

$\Rightarrow$ realizable

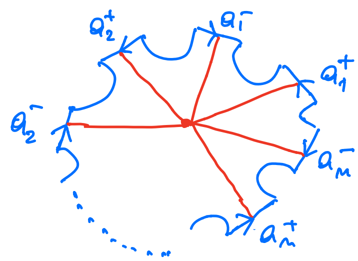
- $\tilde{\Sigma} = S$, $n \geq d$ $\Longrightarrow$ realizable [Bar 01]
- $\tilde{\Sigma} = S$, $d_{ij} \leq 2$, $\ell_i \geq d - \sqrt{d/2}$ [Bar 01]
  $\Longrightarrow$ realizable
- Some cases with [Ger 87]
  $\tilde{\Sigma} = S$, $n = 3$, $\pi_j = [a_j, \dots, a_j, 1]$ [PaPe 09]
- $\tilde{\Sigma} = S$, $n = 3$, $\pi_3 = (k, 1, \dots, 1)$ [Boc 82]
  $x = GCD(d_{ij} : i = 1,2 \quad j = 1, \dots, \ell_i)$ [Zan 95]
  realizable $\Longleftrightarrow$ $k \leq \frac{d}{x}$ [Pak-Zvo 14]
- $\tilde{\Sigma} \neq S$, $n = 3$, $\pi_3 = (k, 1, \dots, 1)$ realizable [Boc 82]
- $\tilde{\Sigma} = S$, $\forall n$, $\pi = (k_i, 1, \dots, 1)$ $i \geq 3$ [SX16]
  $x = GCD(d_{ij} : i = 1,2 \quad j = 1, \dots, \ell_i)$
  realizable $\Longleftrightarrow$ $k_i \leq \frac{d}{x}$ $i \geq 3$
- $(T, S, d, 3; [3,5,4,\dots,4], [4,\dots,4], [2,\dots,2])$ [Izm 13]
  $(T, S, d, 3; [2,4,3,\dots,3], [3,\dots,3], [3,\dots,3])$ [CoZa 18]
  ... exceptional [FePe 18]

* Series of papers by Gonçalves, Zieschang and followers [BGKZ03]
on indecomposable f's (not $f = g \circ h$)

* Some attention to simple data all $\pi_i = [2,1,\dots,1]$ (uniqueness) [BeEd84]

Zheng's computational approach [Zh06]

gluings of d copies of

deg-d branched covers $\longleftrightarrow$  $\longleftrightarrow$ fat graphs

→ count numbers of fat graphs and arrange in generating function
→ reduce to possibly disconnected
→ express coefficients using characters of rep's of $\mathfrak{S}_d$
→ reduce to computation of $p_\sigma(z) \in \mathbb{C}[z]$, $q_\sigma(z) \in \mathbb{C}(z)$, $\sigma \in \mathfrak{S}_d$

- classification of all realizable/exceptional for $d \leq 20$
- conjectures for infinite families later proved true

Sample of counting results

[Pe 19 a/b/c]

γ_V computed exactly for

$$\left(gT, S, 2k, 3; [2,\dots,2], [2h+1,1,2,\dots,2], \pi_3\right)$$

$$\left(gT, S, 2k, 3; [2,\dots,2], [2h+1,3,2,\dots,2], \pi_3\right)$$

$$\left(gT, S, 2k+1, 3; [1,2,\dots,2], [2h+1,2,\dots,2], \pi_3\right)$$

for small g, h and arbitrary k

- $$\nu_V\left(S, S, 2k+1, 3; [1,2,\dots,2], [5,2,\dots,2], [p,q,r]\right) = \begin{cases} 0 & p=q=r \\ 1 & p=q\neq r \\ 2 & p\neq q\neq r \quad p>k \\ 3 & p\neq q\neq r<k \end{cases}$$

- $$\gamma_V\left(2T, S, 2k+1, 3; [1,2,\dots,2], [9,2,\dots,2], [2k+1]\right)$$
$$= \frac{k}{16}\left(7k^3-42k^2+72k-37\right) + \frac{5}{8}(2k-3)\left[\frac{k}{2}\right]$$
(except $\gamma = 10$ for $k=8$)

Special families with many 2's [PePe06]

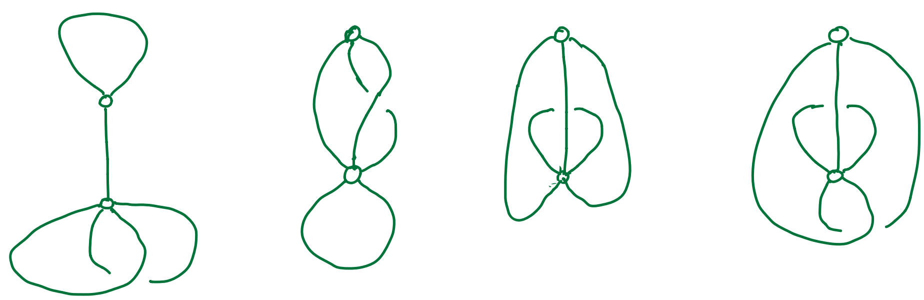
$$\left(T, S, d, 3;\ [2,\dots,2], [5,3,2,\dots,2], [p,q]\right)$$

realizable iff $p \neq q$.

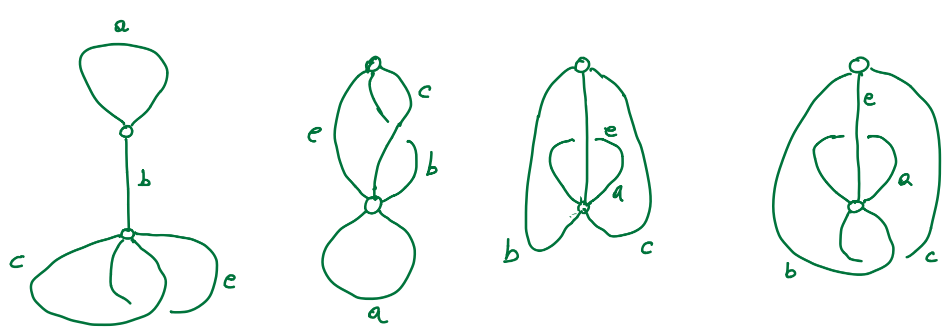
Proof: $[2,\dots,2]$ $[5,3,2,\dots,2]$

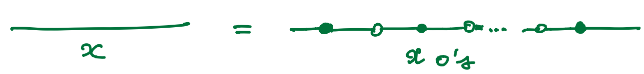
Ignoring 2's: abstractly

in T:



restoring 2's:





realizable: $[a+1, \dots]$ $[b+c+2, \dots]$ $[a+b+c+3, \dots]$

$a+b+c+e = d/2 - 4 \Longrightarrow \dots$ ▨

• $\left(S, S, d, 3\,;\, [2,\dots,2], [5,3,2,\dots,2], [r,s,t,u]\right)$

realizable iff $\pi_4 \neq [r,r,s,s]$ and $\pi_4 \neq [r,r,r,3r]$

• $\left(S, S, d, 3\,;\, [2,\dots,2][3,3,2,\dots,2], [r,s,t]\right)$

realizable iff $r \neq d/2$

Exceptionality and realizability from decomposition [PePe06]

* Show some branch data force any realization f to be $f = g \circ h$ with g realizing non-realizable.

* Show some branch data can be realized by $f = g \circ h$ with g existing by [EKS84] and easy h.

Thm 1: $(S, S, d, m; \pi)$ realizable

$\exists k > 1$ s.t. $k \mid d_{ij}$ $i = 1,2$ $\forall j$

$\Longrightarrow d_{ij} \leq d/k$ $\forall i \geq 3$ $\forall j$

Thm 2: $(S, S, d, m; \pi)$ realizable

$2 \mid d_{ij} \quad i = 1,2 \quad \forall j$

$\Rightarrow \pi_i = [\pi_i', \pi_i'']$ partitions of $d/2$ $\forall i \geq 3$

Thm 3: $(S, S, d, m; \pi)$ realizable

$\exists k > 1$ s.t. $2k \mid d \quad k \mid d_{1j} \quad \forall j$

$2 \mid d_{ij} \quad i = 2,3 \quad \forall j$

$\Rightarrow d_{ij} \leq d/k \quad i = 2,3 \quad \forall j$

$d_{ij} \leq d/2k \quad i \geq 4 \quad \forall j$

Thm 4: $(\tilde{\Sigma}, S, d, 3; \pi)$

$\exists k \geq 3$ s.t. $k \mid d_{ij} \quad \forall i \ \forall j$

$\Rightarrow$ realizable

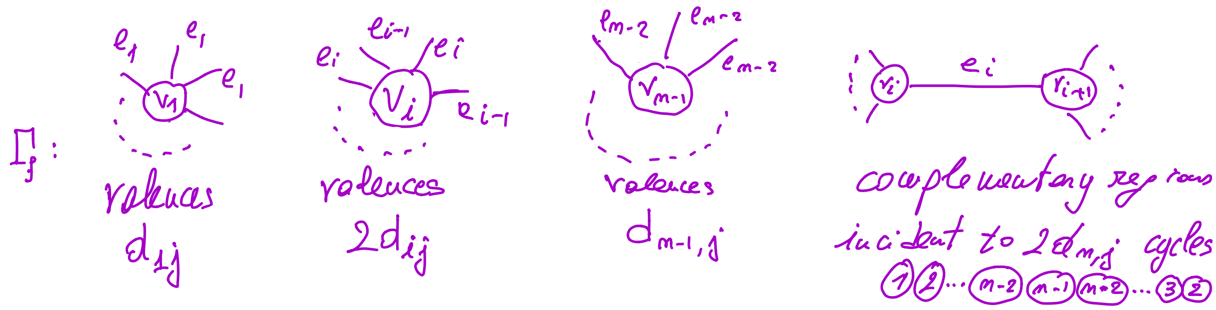
(Rem: short partitions $\Rightarrow$ high genus of $\tilde{\Sigma}$)

Sketch of proof of Thm 2:

I. (First) generalization of DE for $n \geq 4$.

Assuming $f : \tilde{\Sigma} \to S$ exists, lift

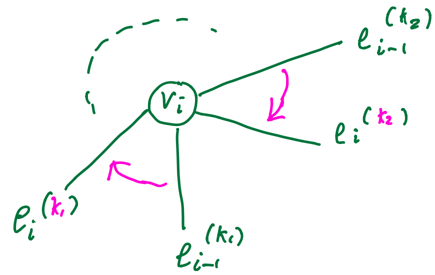
$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \cdots \xrightarrow{e_{n-2}} v_{n-1}$ getting



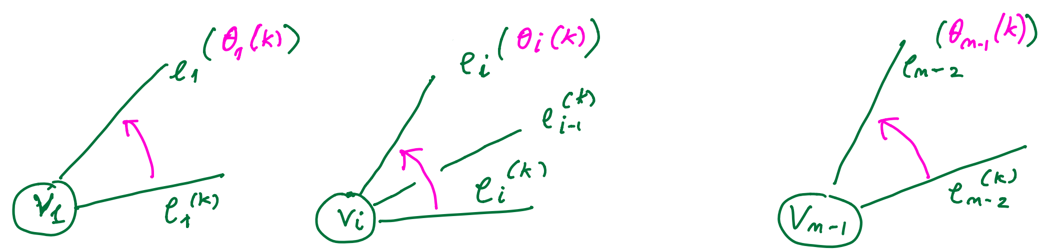
Conversely, any such Γ gives f.

How to read $\theta_1, \dots, \theta_m$ from Γ:

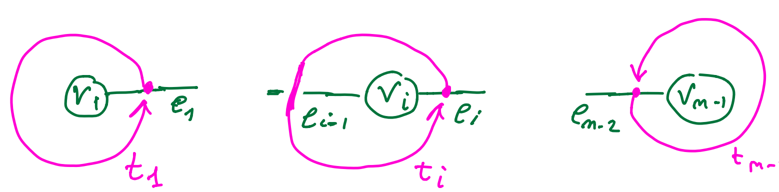
- Number $e_1^{(1)}, \dots, e_1^{(d)}$ randomly
- Number $e_i^{(1)}, \dots, e_i^{(d)}$ recursively so that



- Define θ_i by



because θ_i lifts



Def: k-block decomposition of $\gamma \in \mathfrak{S}_d$ is $\{1,\dots,d\} = \bigsqcup_{m=1}^{d/k} B_m$ s.t. $\# B_m \equiv k$ and $\gamma(B_m) = B_{\hat{\gamma}(m)}$.

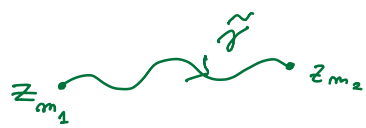
Prop: $f : \tilde{\Sigma} \xrightarrow{d:1} \Sigma$ determined by $\rho : \pi_1(\Sigma^\bullet) \longrightarrow \mathfrak{S}_d$;

$$\exists \quad \begin{tikzcd} \tilde{\Sigma} \arrow[r, "g"', "k:1"] \arrow[dr, "f"'] & \Delta \arrow[d, "h"] \\ & \Sigma \end{tikzcd} \quad \Longleftrightarrow \quad \begin{array}{l} \exists\ k\text{-block decomposition} \\ \text{for all } \gamma \in \operatorname{Im}(\rho) \end{array}$$

Proof: $\Rightarrow$ If $h^{-1}(x_0) = \{z_1, \dots, z_{d/k}\}$ then

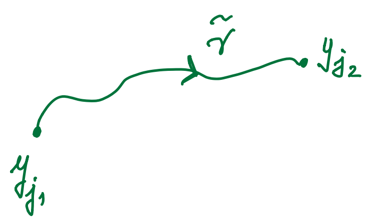
$$f^{-1}(x_0) = \bigsqcup_{m=1}^{d/k} g^{-1}(z_m) \qquad g^{-1}(z_m) = \{ y_j : j \in B_m \}$$

If $\gamma \in \pi_1(\Sigma^\bullet)$ and



in Δ

then



$j_1 \in B_{m_1}$
$j_2 \in B_{m_2}$

in $\tilde{\Sigma}$

$\Rightarrow \rho(\gamma)(B_{m_1}) = B_{m_2}$.

$\Leftarrow$ The action of $\pi_1(\Sigma^\bullet)$ is well-defined on the fibers of $f^\bullet : \tilde{\Sigma}^\bullet \longrightarrow \Sigma^\bullet$ up to conjugation, but block decomposition survives conjugation, so we can define g by collapsing each block to a point. $\square$

Thm 2 direct consequence of:

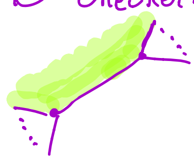
Prop: if $2 \mid d_{ij}$ $i=1,2$ $\forall j$ then any f realizing $(S, S, d, m ; \pi)$ is $h \circ g$ realizing

$$\left(S, S, d/2, 2m-2 ; [d_{1j}/2], [d_{2j}/2], \pi_3', \pi_3'', \dots, \pi_m', \pi_m''\right)$$

$$\left(S, S, 2, m ; [2], [2], [1,1], \dots, [1,1]\right) = \left(S, S, 2, 2 ; [2,2], [2,2]\right)$$

$(g(z) = z^2 : \hat{\mathbb{C}} \to \hat{\mathbb{C}})$

Def: $\Gamma \subset S$ checkerboard graph if $S \setminus \Gamma$ = colored/white discs s.t.



Lem: Γ checkerboard $\Longleftrightarrow$ all vertices even valence.

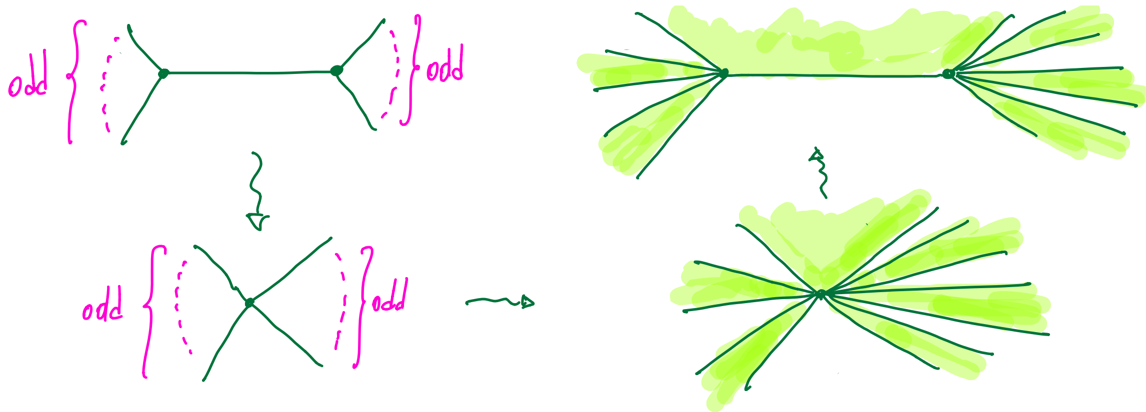
Proof $\Longrightarrow$ obvious

$\Longleftarrow$ induction on # vertices $=: v$

$v = 1 \Longrightarrow$ wedge of circles

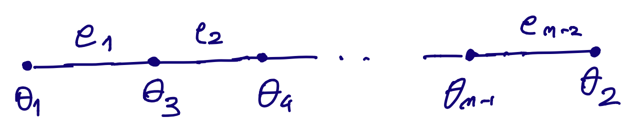
$v > 1$: choose

, contract, apply induction, use assumption:

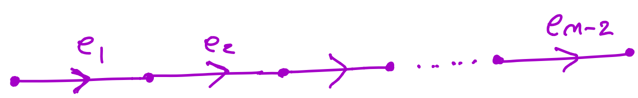


$\blacksquare$

<u>Proof of Prop</u>: enough to show $\{1,\dots,d\}$ splits into two blocks of $d/2$ elements such that θ_1, θ_2 switch them and $\theta_3, \dots, \theta_m$ fix them.

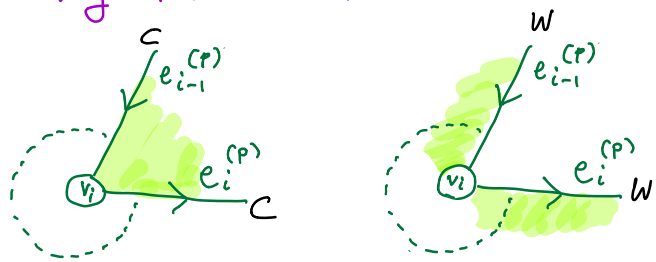
We take 

$\Rightarrow$ Γ has all even valences $\Rightarrow$ checkerboard.

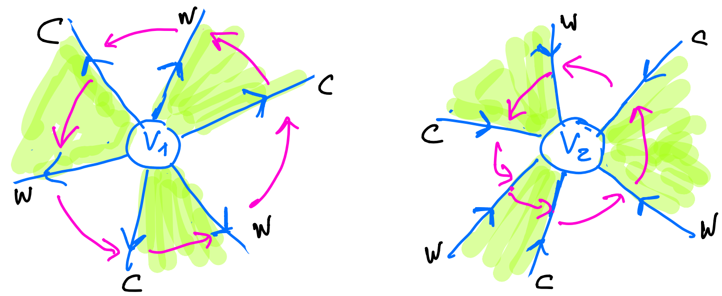
We orient edges 

and lifts accordingly; we declare $p \in \{1,\dots,d\}$ belongs to the colored/white block if $e_i^{(p)}$ has orientation induced by the colored/white incident disc.

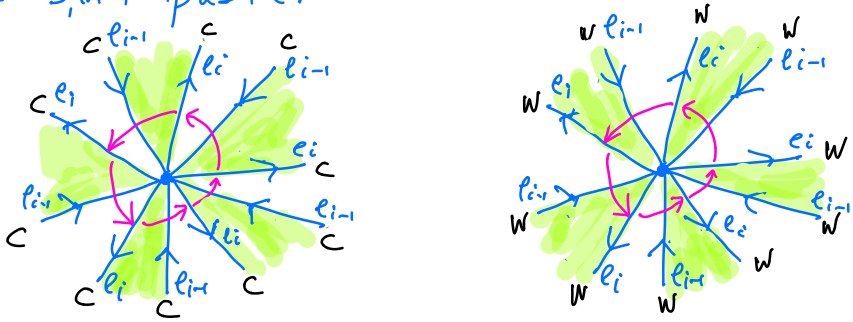
Well-defined indep. of i:



θ_1, θ_2 switch



$\theta_i \quad i = 3, n-1$ preserve.



$\theta_n = (\theta_1 \theta_2 \theta_3 \cdots \theta_{n-1})^{-1}$ preserves. ∎

Checkerboard graphs

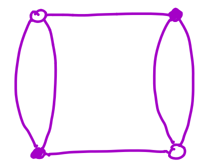
This is shown extending techniques from [Bar01]

Thm: $(\tilde{\Sigma}, S, d, 3\ ;\ \pi_1, \pi_2, [d-2,2])$ realizable except $(T, S, 6, 3\ ;\ [3,3], [3,3], [4,2])$.

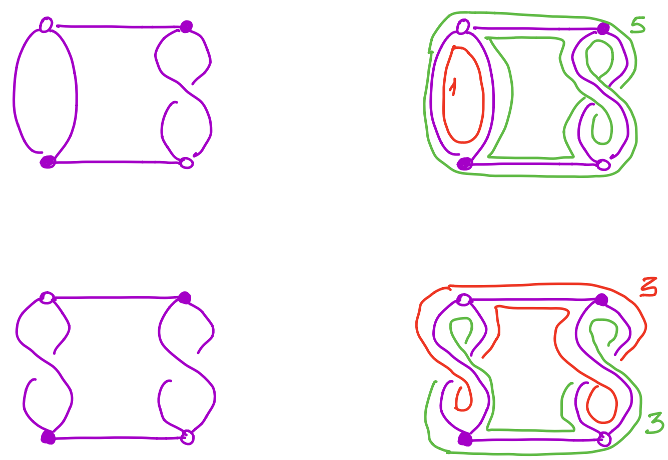
[PePe 08]

Exception from DE: [3,3] [3,3]

abstractly only

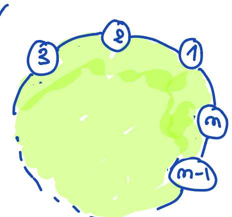


embeddings:



Construction from checkerboard graphs:

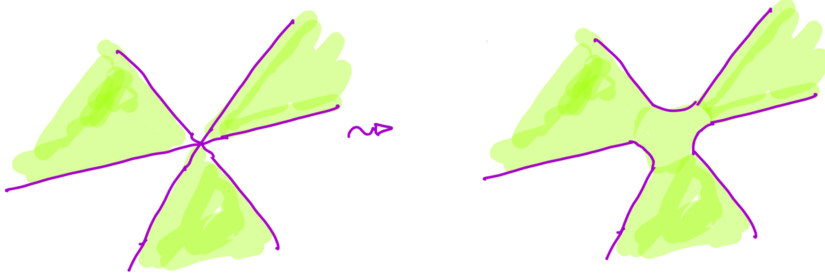
Let $f : \tilde{\Sigma} \to S$ realize and take

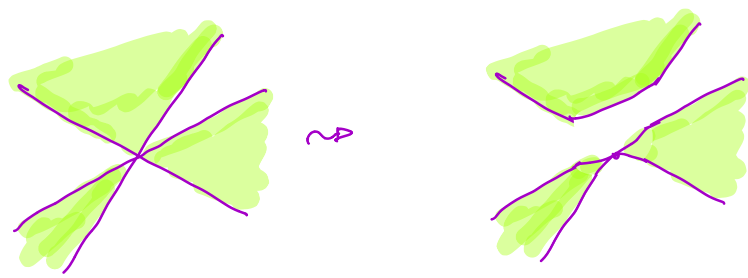
$f^{-1}\Bigg($  $\Bigg)$

getting

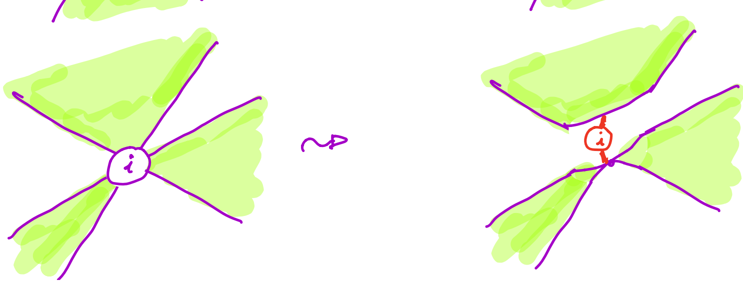
graph Γ with vertices (i) of valence $2d_{ij}$, d colored and d white discs with $(1), (2), (3), \ldots, (n)$ pos/neg arranged on ∂.

Now we merge together distinct colored discs and distinct white discs as long as possible, i.e. until there is one of each type



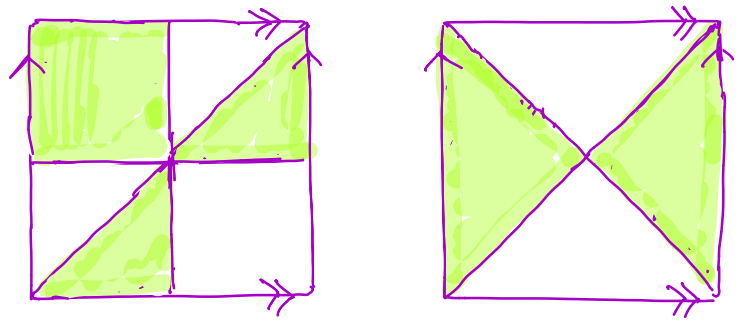


but we keep track of these moves adding trees!

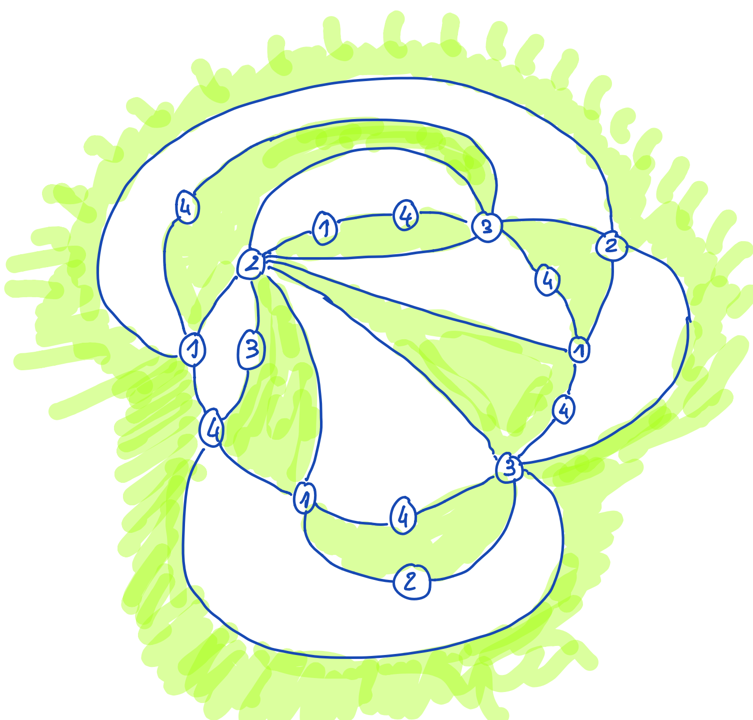


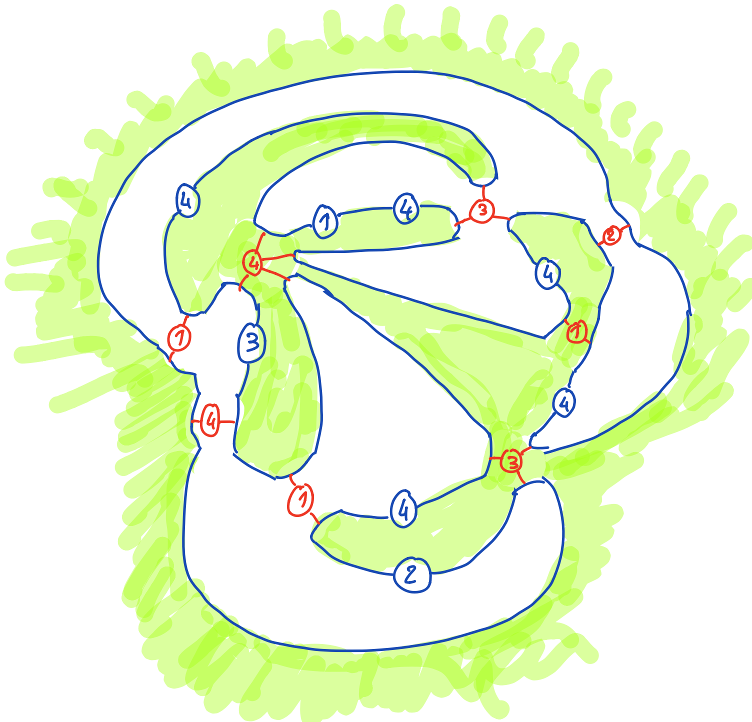
Eventually we get a "minimal checkerboard graph" with labeled vertices and attached trees.

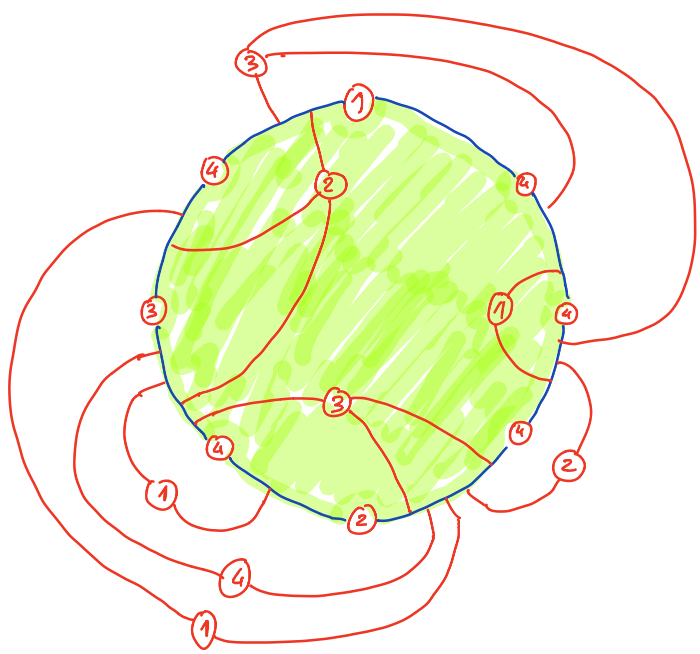
For $\tilde{\Sigma} = S$ this is just a circle; for $\tilde{\Sigma} = T$



<u>Example</u>: realization of $(S, S, 7, 4;\ [2221][421][331][211111])$







- ▷ $(\tilde{\Sigma}, \tilde{S}, d, m; \pi)$ realizable iff $\exists$ minimal checkerboard graph in $\tilde{\Sigma}$ s.t ....

- ▷ construction of all minimal checkerboard graphs

- ▷ decorations realizing all $(\tilde{\Sigma}, S, d, 3; \pi_1, \pi_2, [d-2, 2])$.