Intro to Whitney towers III:
2-spheres in 4-manifolds

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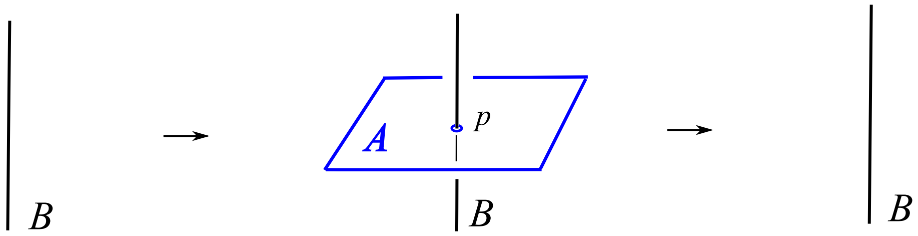
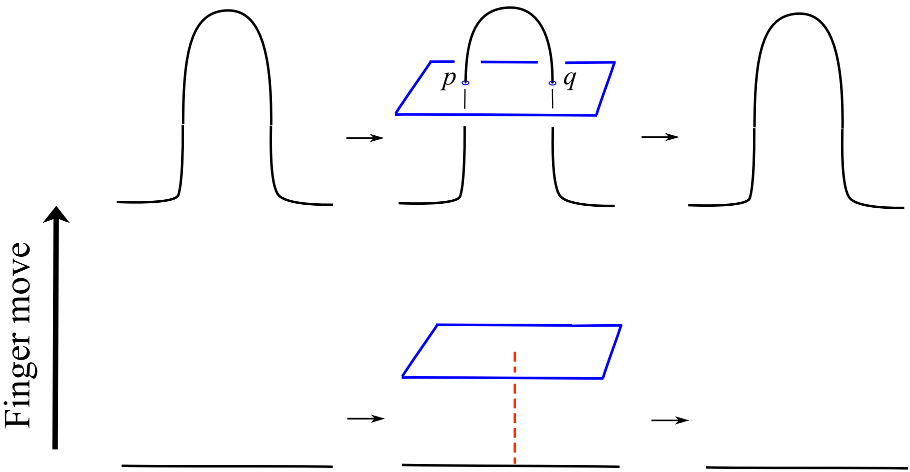
Outline of this talk

- Order 0 intersection form, pulling apart pairs of 2-spheres
- Order 1 intersection invariants, pulling apart triples of 2-spheres, stable embedding of m-tuples of 2-spheres
- Questions

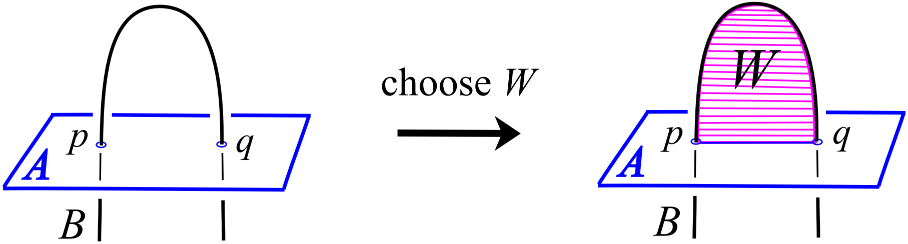
Surface sheets A and B in $B^4 = B^3 \times I$ with $p = A \pitchfork B$ and $A \subset B^3 \times *$



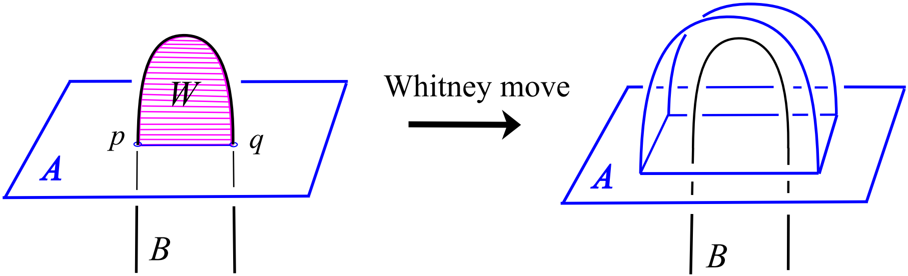
***Finger move*: Before and after**



Whitney disk W pairing $p, q \in A \pitchfork B$

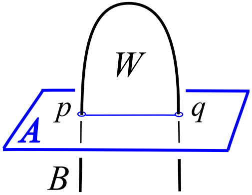


Before and after a Whitney move



Whitney disks in 4-manifolds

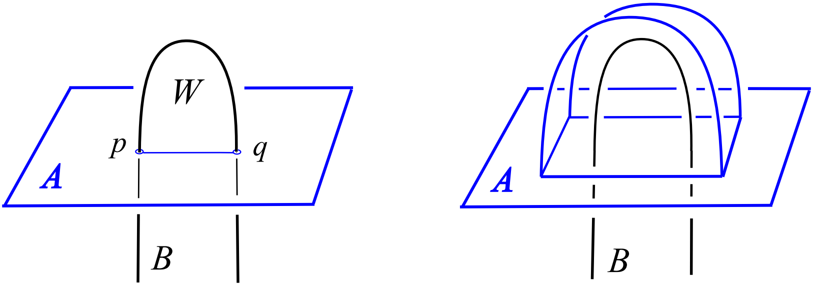
Have just seen a model Whitney disk W pairing $p, q \in A \pitchfork B$ in B^4:



Definition:
A Whitney disk pairing $p, q \in A \pitchfork B$ in a 4-manifold X^4 is diffeomorphic to the model near ∂W, and only differs away from ∂W by *plumbings* in the interior of W.

‘Successful’ Whitney move: W is ‘clean’ and ‘framed’

Eliminates $p, q \in A \pitchfork B$ without creating new intersections in A or B:



Uses: W is ‘clean’ = embedded & interior disjoint from all surfaces.
Uses: W is *framed* = W has appropriate disjoint parallels.

Homotopy of surfaces in 4-manifolds

Regular homotopy =

isotopies + finger moves + (clean, framed) Whitney moves.

Arbitrary homotopy =

regular homotopy + local *cusp homotopies*.

Fundamental question:

"Given $A^2 \looparrowright X^4$, is A homotopic to an embedding?"

First obstructions to making components disjointly embedded:

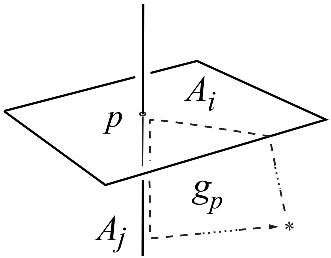
The intersection invariants $\lambda(A_i, A_j) \in \mathbb{Z}[\pi_1 X]$

The self-intersection invariants $\mu(A_i) \in \mathbb{Z}[\pi_1 X]/\text{relations}$

In higher dimensions these obstructions are complete!

Intersection and Self-intersection invariants λ, μ for $A = \cup_i S^2 \overset{A_i}{\looparrowright} X^4$

$$\lambda(A_i, A_j) := \sum_{p \in A_i \pitchfork A_j} \epsilon_p \cdot g_p \in \mathbb{Z}[\pi_1 X]$$



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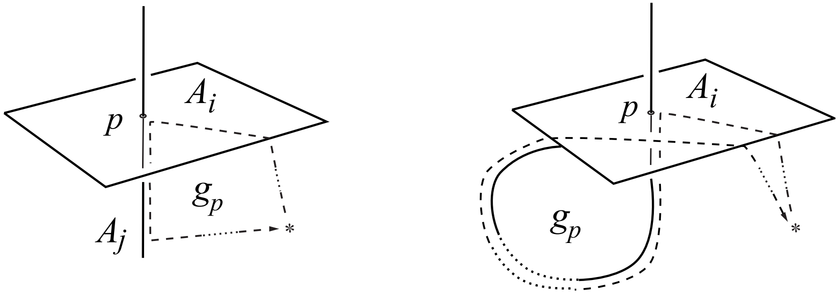
$$\lambda(A_i, A_j) := \sum_{p \in A_i \pitchfork A_j} \epsilon_p \cdot g_p \in \mathbb{Z}[\pi_1 X]$$

and

$$\mu(A_i) := \sum_{p \in A_i \pitchfork A_i} \epsilon_p \cdot g_p \in \frac{\mathbb{Z}[\pi_1 X]}{\mathbb{Z}[1] \oplus \langle g - g^{-1} \rangle}.$$



Relations in target $\frac{\mathbb{Z}[\pi_1 X]}{\mathbb{Z}[1] \oplus \langle g - g^{-1} \rangle}$ of the self-intersection invariant μ:

- $g - g^{-1} = 0$ accounts for choice of orientation on loop determining $g_p \in \pi_1 X$ for self-intersections $p \in A_i \pitchfork A_i$.

- $1 = 0$ accounts for cusp homotopies of A_i creating/eliminating self-intersections $p \in A_i \pitchfork A_i$ with trivial $g_p = 1 \in \pi_1 X$.

λ and μ are invariant under homotopies of A
(isotopies, finger moves, Whitney moves, cusp homotopies).

Can express λ and μ as sums of *decorated order zero trees*:

$$\lambda(A_i, A_j) = \sum_{p \in A_i \pitchfork A_j} \epsilon_p \cdot \; i \xrightarrow{\quad g_p \quad} j \quad \text{for } i \neq j$$

and

$$\mu(A_i) = \sum_{p \in A_i \pitchfork A_i} \epsilon_p \cdot \; i \xrightarrow{\quad g_p \quad} i$$

modulo relations:

$$i \xrightarrow{\quad g_p \quad} i = i \xrightarrow{\quad g_p^{-1} \quad} i \quad \text{and} \quad i \xrightarrow{\quad 1 \quad} i = 0$$

So these classical intersection invariants $\lambda_0 := \lambda$ and $\mu_0 := \mu$ can be expressed as a single order 0 'tree-valued' invariant:

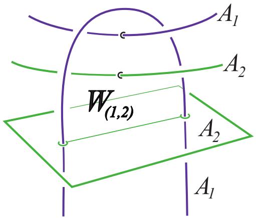
$$\tau_0(A) := \sum_{p \in A_i \pitchfork A_j} \epsilon_p \cdot \, i \xrightarrow{g_p} j$$

modulo relations:

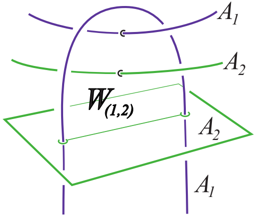
$$i \xrightarrow{g_p} j = i \xleftarrow{g_p^{-1}} j \quad \text{and} \quad i \xrightarrow{1} i = 0$$

Before generalizing $\tau_0 = 0 \rightsquigarrow \tau_1$, will consider $\lambda_0 = 0 \rightsquigarrow \lambda_1$...

$\lambda_0(A_1, A_2) = 0 \rightsquigarrow$ Whitney disks pairing $A_1 \cap A_2$:



$\lambda_0(A_1, A_2) = 0 \rightsquigarrow$ Whitney disks pairing $A_1 \cap A_2 \overset{\text{htpy}}{\rightsquigarrow} A_1 \cap A_2 = \emptyset$:



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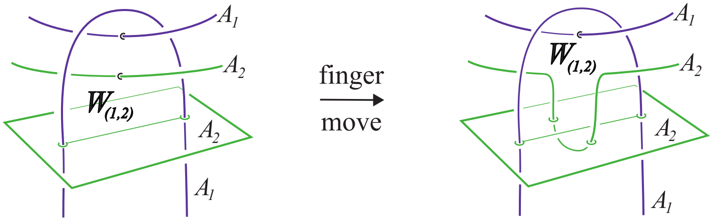

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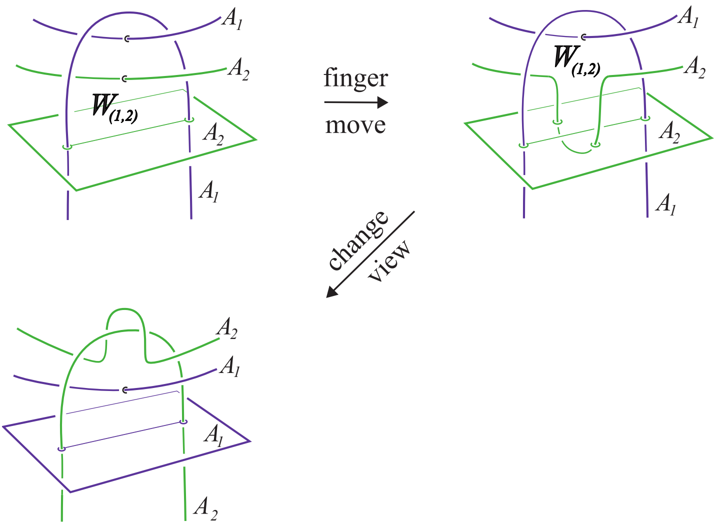

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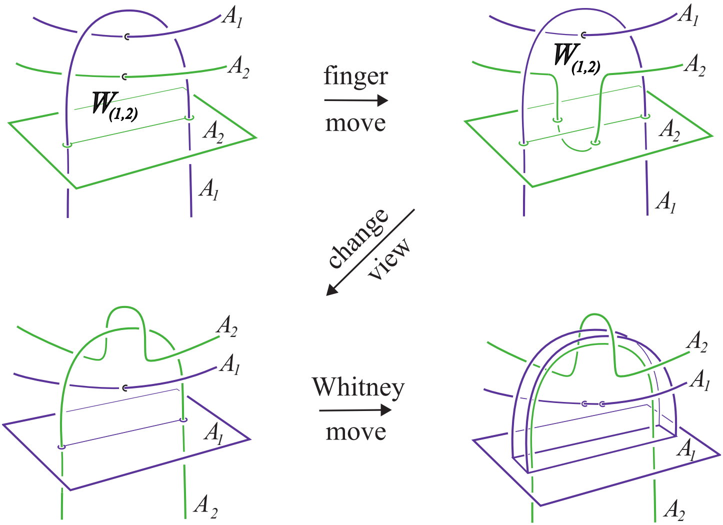
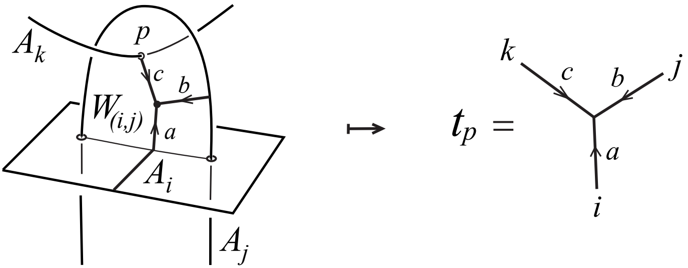


Fails for ≥ 3 components: How to eliminate $W_{(1,2)} \cap A_3$??

Will generalize $\lambda_0(A_1, A_2)$ to $\lambda_1(A_1, A_2, A_3)$ counting $W_{(1,2)} \cap A_3$...

$\lambda_0(A_i, A_j) = 0 \Leftrightarrow \exists$ Whitney disks $W_{(i,j)}$ pairing all $A_i \cap A_j$.



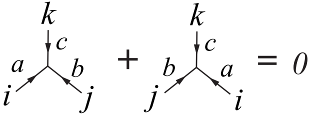
$a, b, c \in \pi_1 X$

$$\lambda_1(A_1, A_2, A_3) := \sum \epsilon_p \cdot t_p \in \frac{\langle \pi_1 X\text{-decorated order 1 Y-trees} \rangle}{\text{AS, HOL and INT relations}}$$

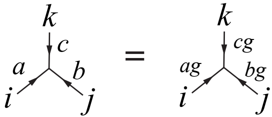
sum over $p \in W_{(i,j)} \cap A_k$ for $i < j < k$ (cyclic ordering).

The *Antisymmetry* and *Holonomy* relations:

AS:



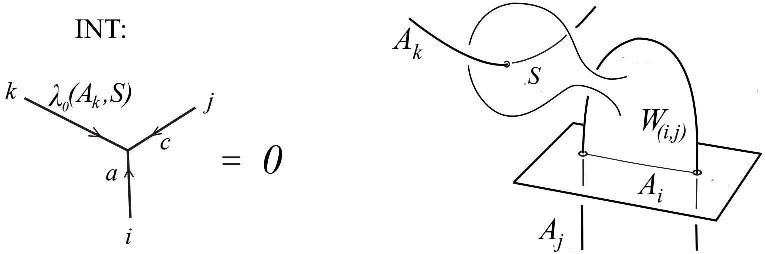
HOL:



The AS relations make signs well-defined.

The HOL relations account for whisker choices on the Whitney disks.

The INT *Intersection* relations depend on A and $\pi_2 X$ via λ_0:



over $S : S^2 \to X$ representing generators for $\pi_2(X)$.

The INT relations account for choices of the interiors of Whitney disks.

Theorem:

1. *$\lambda_1(A_1, A_2, A_3)$ only depends on the homotopy classes of the A_i.*

2. *$\lambda_1(A_1, A_2, A_3)$ vanishes if and only if A_1, A_2, A_3 can be made pairwise disjoint by a homotopy.*

3. *$\lambda_1(A_1, A_2, A_3)$ vanishes if and only if $A_1 \cup A_2 \cup A_3$ admits an order 2 non-repeating Whitney tower:*
   All $W_{(i,j)} \pitchfork A_k$ paired by $W_{((i,j),k)}$ for distinct i, j, k.

Open Problem:

Show that the order 2 invariant $\lambda_2(A_1, A_2, A_3, A_4)$ is well-defined... so far only partial progress.

2-sphere $A : S^2 \looparrowright X^4$, $\mu_0(A) = 0 \rightsquigarrow$ framed W_r pairing $A \pitchfork A$.

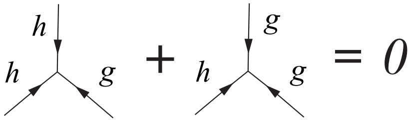
As before, $p \in W_r \pitchfork A \mapsto \pi_1 X$-decorated Y-tree t_p.

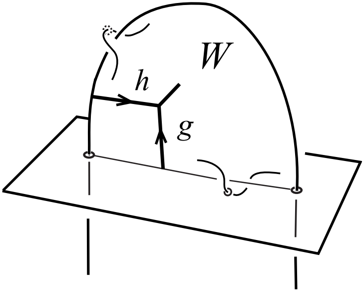
$$\tau_1(A) := \sum \epsilon_p \cdot t_p \in \frac{\langle \pi_1 X\text{-decorated order 1 Y-trees} \rangle}{\text{AS, HOL, FR and INT relations}}$$

sum over all $p \in W_r \cap A$.

The new FR *Framing* relations correspond to opposite boundary-twists along different arcs of ∂W:

FR:





Theorem:

1. $\tau_1(A)$ *only depends on the homotopy class of A.*

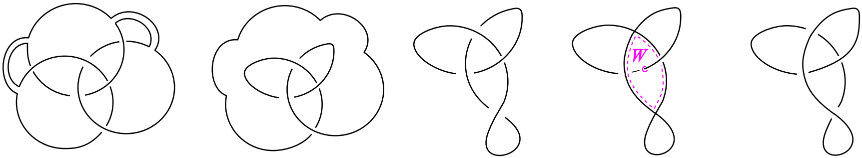
2. $\tau_1(A)$ *vanishes if and only if A admits an* <u>*order 2*</u> *Whitney tower. (Exist framed second order Whitney disks pairing all* $W_r \pitchfork A$*.)*

3. $\tau_1(A)$ *vanishes if and only if A admits a* <u>*height 1*</u> *Whitney tower. (Exist framed* W_r *pairing* $A \pitchfork A$ *which have interiors disjoint from A, but may have* $W_r \pitchfork W_s \neq \emptyset$*.)*

4. $\tau_1(A)$ *vanishes if and only if A is* <u>*stably homotopic*</u> *to an embedding. (A is homotopic to an embedding in* $X \overset{n}{\#} S^2 \times S^2$*.)*

- X simply-connected $\Rightarrow \tau_1(A) \in \mathbb{Z}/2\mathbb{Z}$ or 0.

- Example: $A = 3\mathbb{CP}^1 \looparrowright \mathbb{CP}^2 \Rightarrow \tau_1(A) = 1 \neq 0 \in \mathbb{Z}/2\mathbb{Z}$.



- Quotient of target by $\pi_1 X \to 1 \Rightarrow \tau_1(A) \in \mathbb{Z}/2\mathbb{Z}$ or 0.

- $\lambda_0(A, S) = 1$ for some $S \in \pi_2 X \Rightarrow \tau_1(A) \in \mathbb{Z}/2\mathbb{Z}$ or 0.

In these cases $\tau_1(A) = \mathrm{km}(A)$, the *Kervaire–Milnor* invariant.

Non-trivial $\pi_1 X$ edge decorations can make the target of τ_1 large:

$\pi_1 X$ left-orderable and INT trivial $\Rightarrow \tau_1(A) \in \mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty$.

Can realize values in target of τ_1 in 4-manifolds with non-empty boundary via framed link descriptions.

E.g. Attach a 0-framed 2-handle H to a null-homotopic knot K in ∂(4-ball $\cup$ 1-handles), where K is created by banding together the Borromean rings with bands running around the 1-handles, and take $A = H \cup$ null-homotopy of K.

Open Problem:

Find an example of $A \looparrowright X$, where X is closed and $\tau(A) \neq 0$ after quotient of target which kills the Y-tree with all three edges labelled by the trivial element $1 \in \pi_1 X$.

Even after trivializing all $\pi_1 X$-decorations,
τ_1 sees global information in <u>closed</u> 4-manifolds:

Theorem: (Freedman–Kirby, Kervaire–Milnor, Stong)

Suppose X^4 is closed and $H_2(X;\mathbb{Z}/2\mathbb{Z})$ is spherical.
If $A : S^2 \looparrowright X$ is characteristic and $\mu_0 A = 0$, then

$$(\pi_1 X \to 1): \quad \tau_1 A \quad \mapsto \quad \frac{A \cdot A - \mathit{signature}(X)}{8} \quad \mathit{mod}\ 2$$

Question:
What global info is carried by the π_1-decorations in $\tau_1 A$?

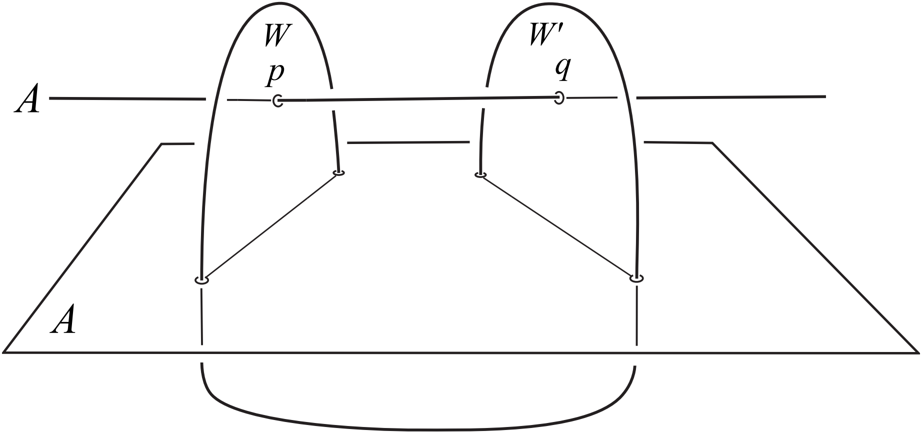
Strategy for proving that $\tau_1(A)$ is a well-defined homotopy invariant:

1. Show that $\tau_1(A)$ does not depend on the choice of $\mathcal{W}$ (Whitney disk interiors, boundaries, pairings of self-intersections and preimages of self-intersections) for a fixed immersion $A \looparrowright X$.

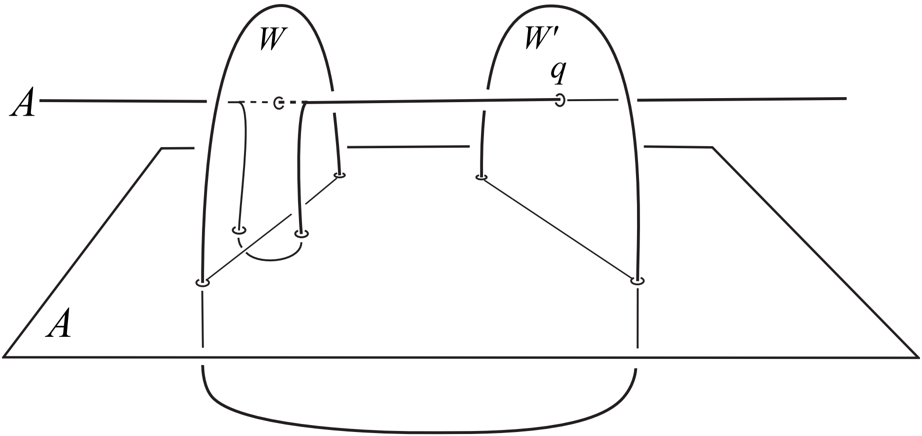
2. Homotopy invariance follows: If A is homotopic to A', then exists A'' which differs from each of A and A' by finger moves which can be made disjoint from all Whitney disks by a small isotopy.

$\tau_1(A) = 0 \rightsquigarrow$ **order 2 $\mathcal{W}$ supported by A**

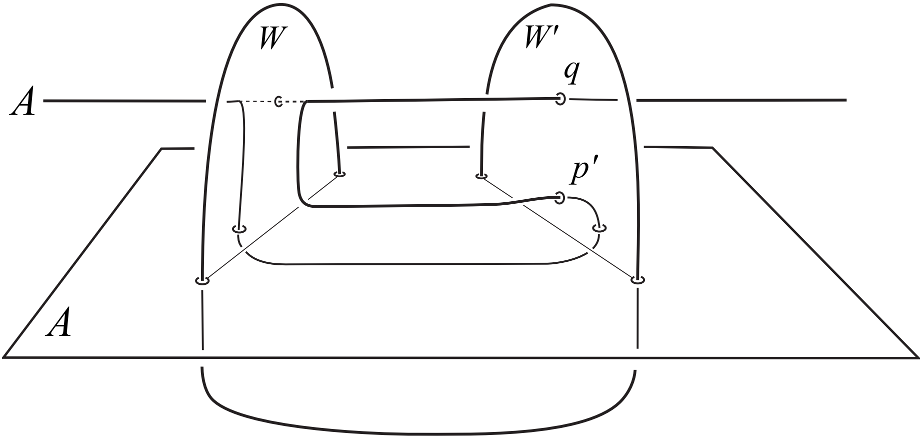
Key step in 'algebraic cancellation' $\Rightarrow$ 'geometric cancelation':
Will 'transfer' p from W to $p' \in W'$ to pair with q.



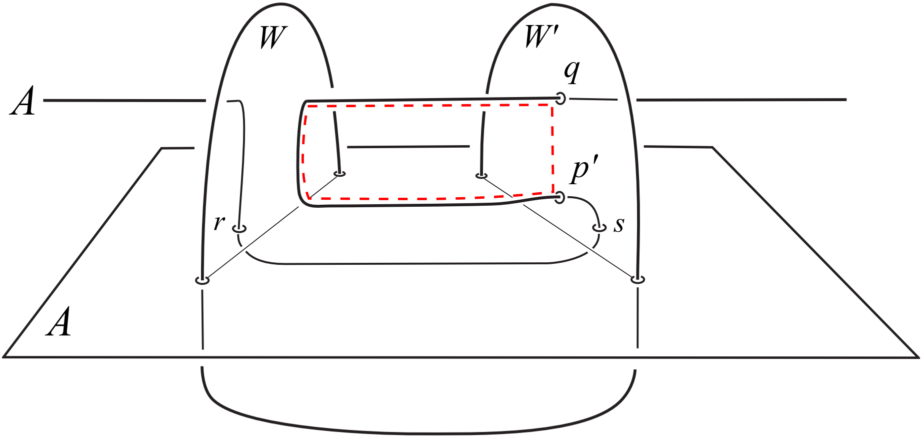
Finger move pushing down along W into A:



Finger move pushing along A:

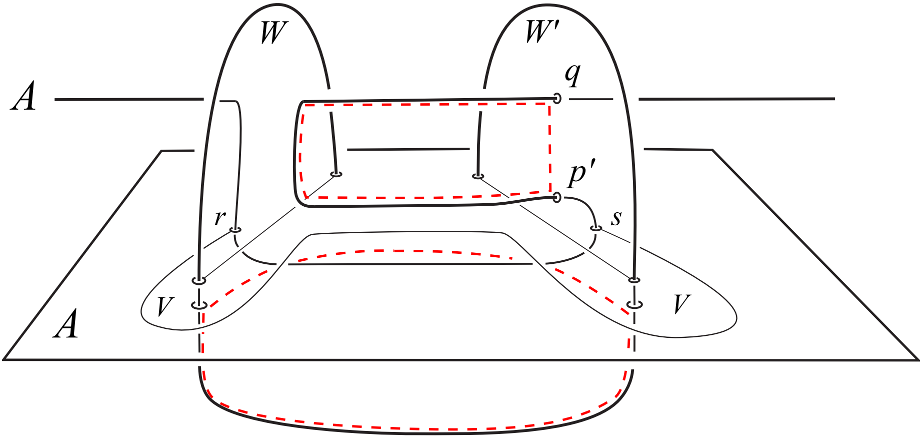


Have $p', q \in W' \pitchfork A$ paired by (uncontrolled) order 2 Whitney disk.



Need to pair $r, s \in A \pitchfork A$.

Can pair $r, s \in A \pitchfork A$ by local order 1 Whitney disk V ('under' horizontal sheet).



Can pair intersections in interior of V by (uncontrolled) order 2 Whitney disk.

Open Problem:

Formulate and prove invariance of a next order $\tau_2(A)$.

Hard part: Showing independence of the choice of boundaries of the order 1 Whitney disks.