# The mysterious geometry of Artin groups
## Talk 1: Among friends

Jon McCammond

UC Santa Barbara

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Caen, France

# Relations, Presentations and Groups

## Definition (Relations)

A braid relation of length $m$ between generators $a$ and $b$ equates the two strictly alternating positive words of length $m$. For $m = 2, 3, 4, \ldots$ these relations are $ab = ba$, $aba = bab$, $abab = baba$, and so on.

## Definition (Presentations)

An Artin presentation has at most one braid relation for each pair of distinct generators and no other relations. Coxeter presentations add relations to make the generators involutions.

## Definition (Groups)

Artin groups are defined by Artin presentations and Coxeter groups are defined by Coxeter presentations

# Conventions for Diagrams

There are two conventions for encoding these presentations as an edge-labeled simple graph with vertices indexing generators and decorated edges indicating braid relations.

| $m$ | Classical | Modern |
|---|---|---|
| 2 | no edge ● ● | no label ●——● |
| 3 | no label ●——● | label $\bullet\overset{3}{\text{———}}\bullet$ |
| $>3$ | label $\bullet\overset{m}{\text{———}}\bullet$ | label $\bullet\overset{m}{\text{———}}\bullet$ |
| $\infty$ | label $\bullet\overset{\infty}{\text{———}}\bullet$ | no edge ● ● |

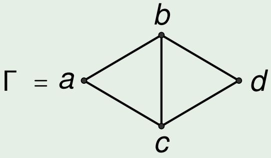
| | Classical | Modern |
|---|---|---|
| disconnected | direct product | free product |
| no labels | small-type | right-angled |

# A small example

Here is a small example in the classical notation.

**Example**

A simple graph Γ and its small-type Artin group $\textsc{Art}(\Gamma)$:



$$\textsc{Art}(\Gamma) = \left\langle a, b, c, d \;\middle|\; \begin{array}{ccc} aba = bab, & ad = da, & bdb = dbd \\ aca = cac, & bcb = cbc, & cdc = dcd \end{array} \right\rangle$$

The corresponding Coxeter group would add the relations $a^2 = b^2 = c^2 = d^2 = 1$.

# Spherical and Euclidean Coxeter groups

The story of these groups is rooted in geometry.

**Remark (Spherical and Euclidean Coxeter groups)**

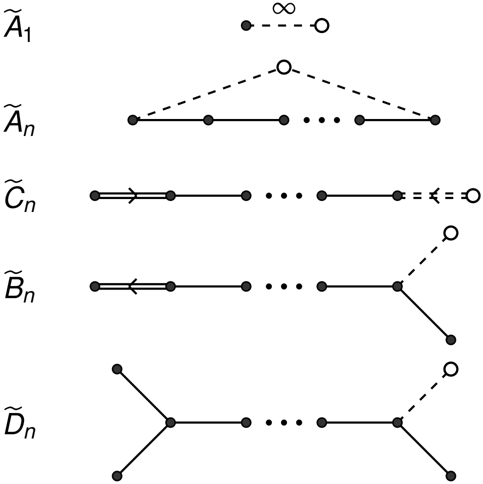
Spherical and Euclidean Coxeter groups are reflection groups that act geometrically on spheres and euclidean space. They arise in the study of regular polytopes and Lie theory. They are the key examples that motivate the general theory introduced by Jacques Tits in the early 1960s.

**Remark (Dynkin diagrams)**

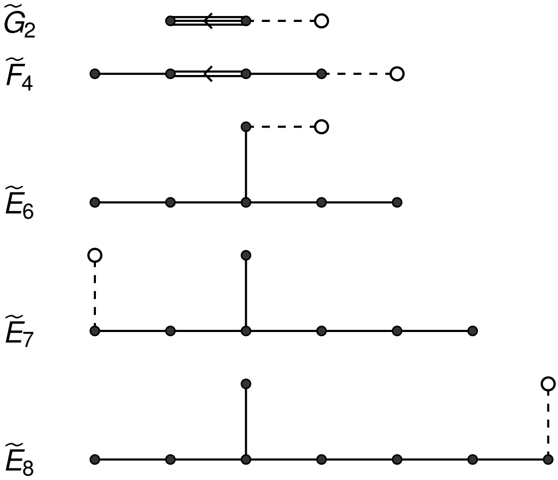
The classification of spherical and euclidean Coxeter groups is classical and their presentations are encoded in the well-known Dynkin diagrams and extended Dynkin diagrams.

The extended Dynkin diagrams consist of:

# Four infinite families



# Five sporadic examples



# Simplicial tilings

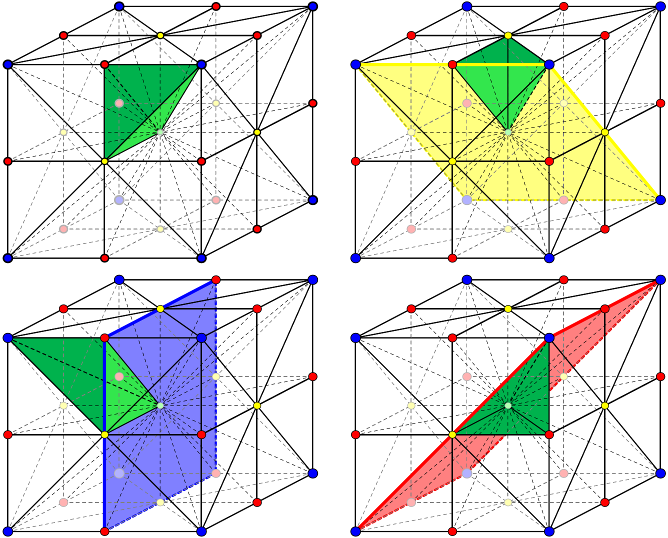
## Remark (Simplices and tilings)

Each (extended) Dynkin diagram with $n+1$ vertices encodes the shape of a simplex $\sigma$ in $\mathbf{S}^n$ or $\mathbf{E}^n$ where every dihedral angle is $\frac{\pi}{k}$ for some positive integer $k > 1$. The images of this chamber $\sigma$ under the group generated by reflections in its facets form a simplicial tiling of $\mathbf{S}^n$ or $\mathbf{E}^n$.

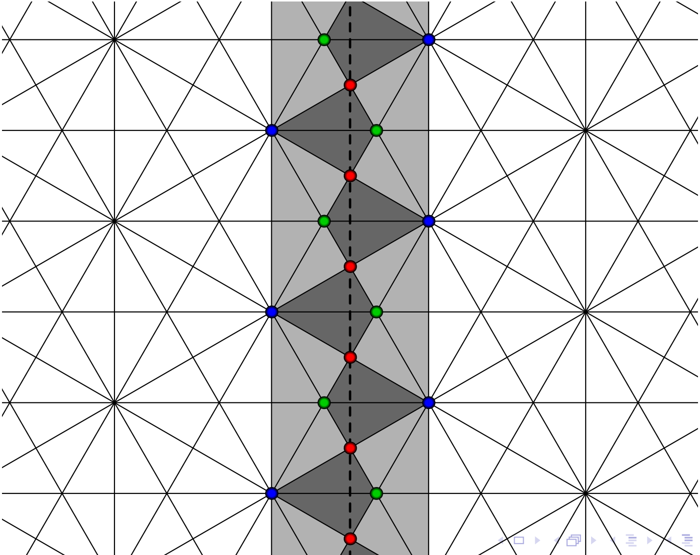
## Remark (Reducible diagrams)

A Coxeter/Artin group is reducible when its defining diagram is disconnected in the classical notaion. The general Coxeter group acting geometrically on $\mathbf{S}^n$ or $\mathbf{E}^n$ is defined by a disjoint union of (extended) Dynkin diagrams. Their fundamental domains are orthogonal spherical joins in the spherical case and metric direct products in the euclidean case.

# The spherical Coxeter group $\mathrm{Cox}(B_3)$



# The euclidean Coxeter Group $\mathrm{Cox}(\widetilde{G}_2)$



# $W$-permutahedra

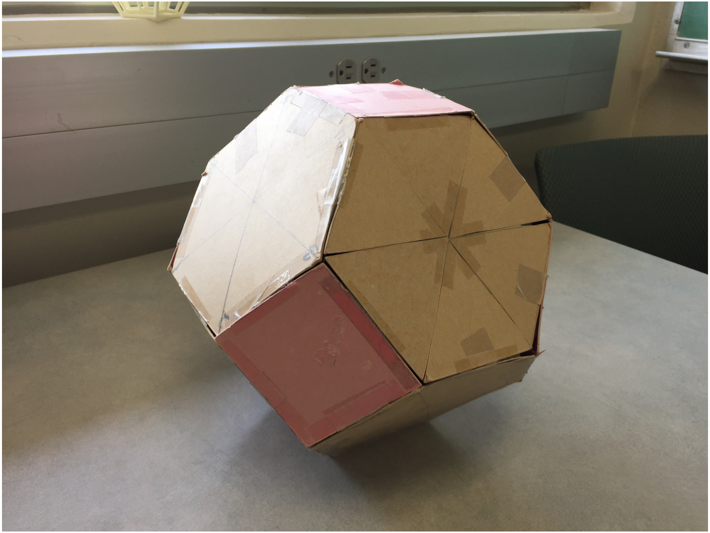
Let $W$ be an irreducible spherical Coxeter group and let $\mathcal{C}$ be a simplicial euclidean cone generated by positive scalar multiples of the points in the spherical simplex $\sigma$.

**Definition ($W$-permutahedra)**

There is a unique point $x$ in the simplicial cone $\mathcal{C}$ that is distance $1/2$ from each of its facets. The convex hull of the $W$-orbit of $x$ is called a (metric) $W$-permutahedron.

When $W$ is a reducible spherical Coxeter group, one takes a direct metric product of the permutahedra for its irreducible components. The name comes from the special case of the symmetric group.

# The $\textsc{Sym}_4$ Permutahedron



# Symmetric bilinear forms

Jacques Tits introduced general Coxeter groups in the early 1960s and proved many facts about them.

## Definition (Coxeter matrix)

Let $W$ be a Coxeter group with generators $s_1, s_2, \ldots, s_n$ and let $m_{ij} = m_{ji}$ be the length of the Artin relation involving $s_i$ and $s_j$. When $i = j$ we define $m_{ii} = 1$ and when there is no relation between $s_i$ and $s_j$ we define $m_{ij} = \infty$. The Coxeter matrix $M$ is the $n$ by $n$ matrix whose $(i, j)$-entry is $\cos(\pi - \frac{\pi}{m_{ij}})$.

## Definition (Symmetric bilinear form)

A Coxeter matrix defines a symmetric bilinear form on $\mathbb{R}^n$ by the formula $\langle u, v\rangle = u^{\mathrm{tr}} M v$ for column vectors $u$ and $v$.

# Linear representations

Let $W$ be a Coxeter group with $n$ generators, let $\mathbb{R}^n$ be a vector space with standard basis $\{e_1, e_2, \ldots, e_n\}$ and let $\langle u, v\rangle$ be the symmetric bilinear form on $\mathbb{R}^n$ defined by the Coxeter matrix $M$.

## Definition (A linear representation)

Tits defined a linear representation for every Coxeter group $W$. The $i$-th generator $s_i$ is sent to the reflection $r_i$ defined by the equation $r_i(v) = v - 2\langle v, e_i\rangle e_i$. This sends $e_i$ to $-e_i$ and it fixes its orthogonal complement. It is easy to check that this is an involution and that the other relations are satisfied.

## Theorem (Tits)

*For every Coxeter group $W$ this representation is faithful.*

# A hyperbolic example: the diagram

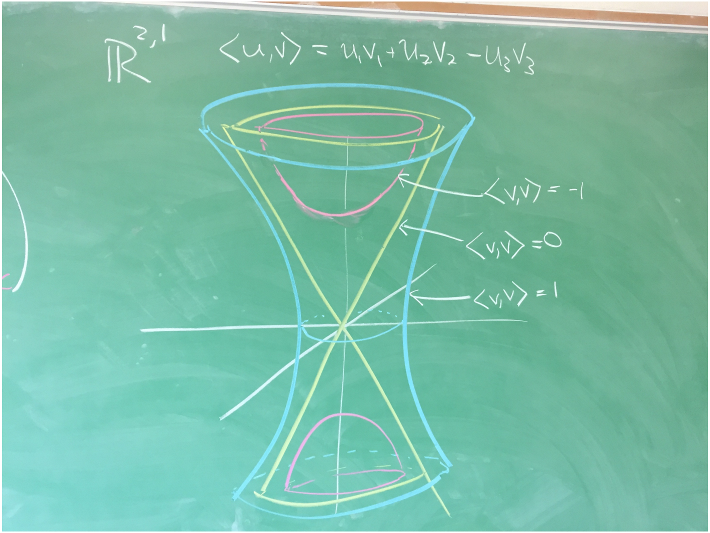
The following small example will be used throughout this talk.

**Example**

The Coxeter group $\langle a, b, c \mid ab = ba, bcb = cbc, a^2 = b^2 = c^2 = 1\rangle$ has Coxeter matrix $M = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 1 \end{bmatrix}$.

This matrix has two positive eigenvalues and one negative eigenvalue. Thus the linear transformations that preserve the bilinear form preserve the surfaces of the form $x^2 + y^2 - z^2 = k$ (in a different basis).

# Lorentzian Geometry



# Tits cone

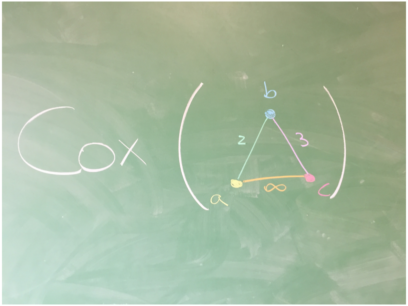
### Definition (Tits cone)

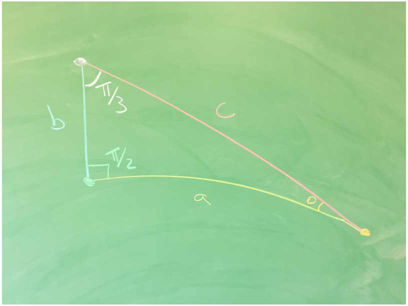
For every Coxeter group, one can use its faithful linear representation to can find a nice space on which it acts. This might be a sphere, euclidean space, hyperbolic space, or more generally the interior of its Tits cone. The Tits cone is the union of the images of a standard simplicial cone $\mathcal{C}$ under the action of the Coxeter group.

### Example

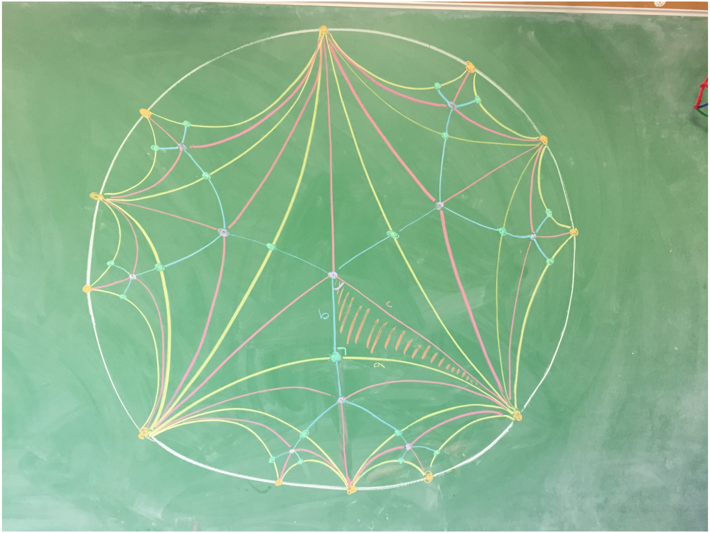
The $(2, 3, \infty)$ Coxeter group acts on $\mathbb{R}^{2,1}$ preserving each component of a hyperboloid of 2-sheets. The interior of its Tits cone is the set of vectors above the positive light cone. In the hyperbolic plane we see the tiling by $\frac{\pi}{2}$, $\frac{\pi}{3}$, 0 triangles.

# A hyperbolic example: the triangle





# A hyperbolic example: the tiling



# The contragradient representation

Note that singular Coxeter forms cause a slight problem.

**Remark (Singular forms)**

When the Coxeter matrix has 0 as an eigenvalue, the hyperplanes orthogonal to the basis vectors $e_i$ do not bound a simplicial cone. Tits' solution is to replace each matrix in the linear representation with its inverse transpose. This contragradient representation has hyperplanes that always bound a simplicial cone and the orbit of this cone is the official definition of the Tits cone.

When $M$ is non-singular the two representations are equivalent and this step is optional.

# The Davis Complex

Mike Davis introduced a very nice complex for each Coxeter group. It has a cell structure dual to the Tits cone.

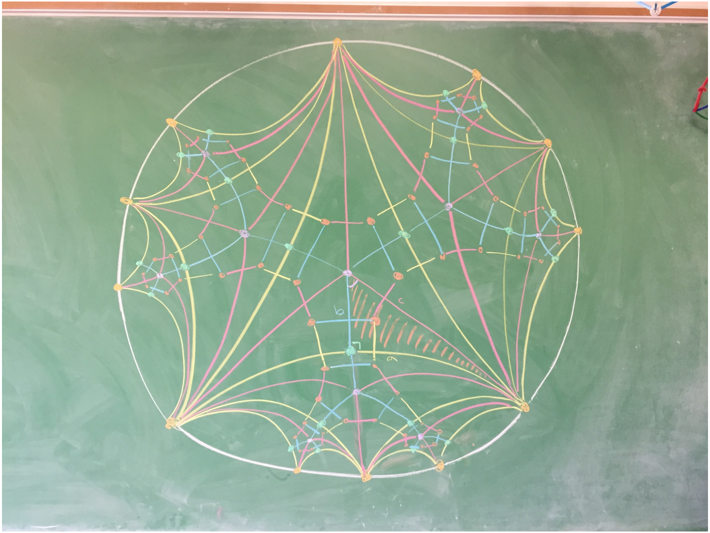
## Definition (Cayley graph)

The unoriented (right) Cayley graph of $W$ with respect to its standard generating set $S$ is the 1-skeleton of the complex dual to the Tits cone.

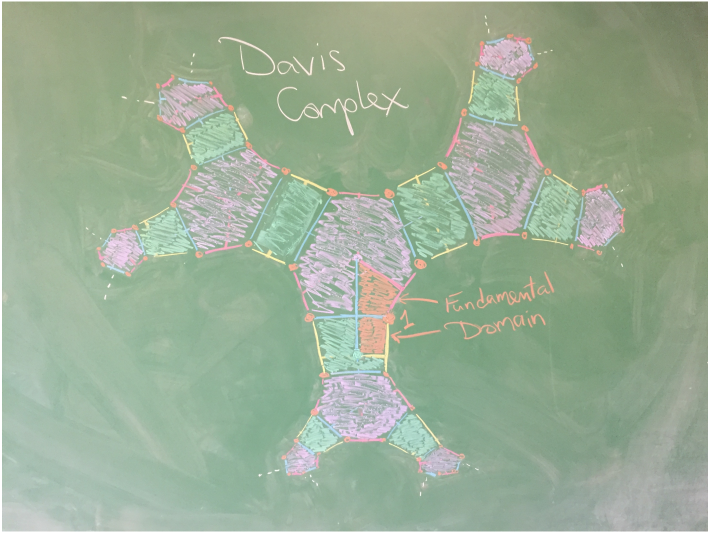
## Definition (Davis complex)

The Davis complex is obtained from the unoriented Cayley graph by attaching permutahedra. For each subset $S' \subset S$ such that the parabolic subgroup $W' = \langle S' \rangle$ is finite we attach a $W'$-permutahedron to each coset $wW'$ in $W$.

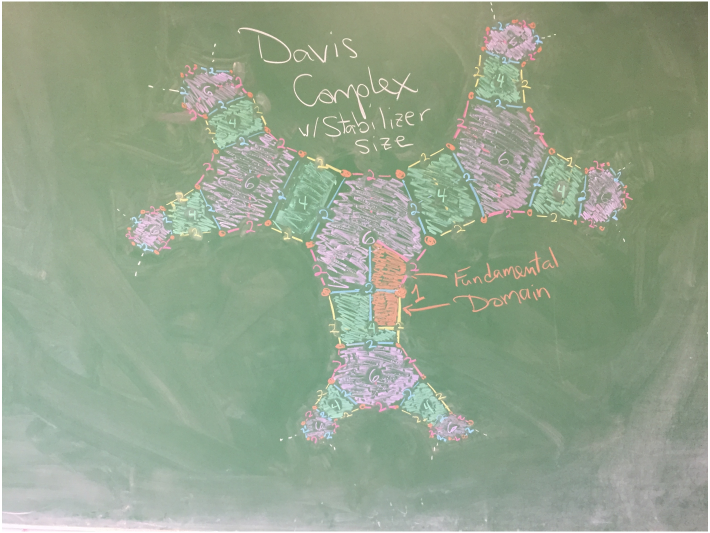
# A hyperbolic example: the Cayley graph



# A hyperbolic example: the Davis complex



# A hyperbolic example: stabilizer size



# Davis complex with the Moussong metric

The piecewise euclidean metric on the Davis complex coming from the metric permutahedra is called the Moussong metric.

**Theorem**

*For every Coxeter group* $W = Cox(\Gamma)$, *the Davis complex with the Moussong metric is CAT(0) and the action of* $W$ *on* $Davis(\Gamma)$ *is geometric (properly discontinuous, cocompact and by isometries).*

Coxeter groups are also known to have many other nice properties. In short, Coxeter groups fit into many of the powerful theories of geometric group theory and are algorithmically very very nice.

# Braid group of a group action

We now transition from Coxeter groups to Artin groups.

**Definition (Braid group of an action)**

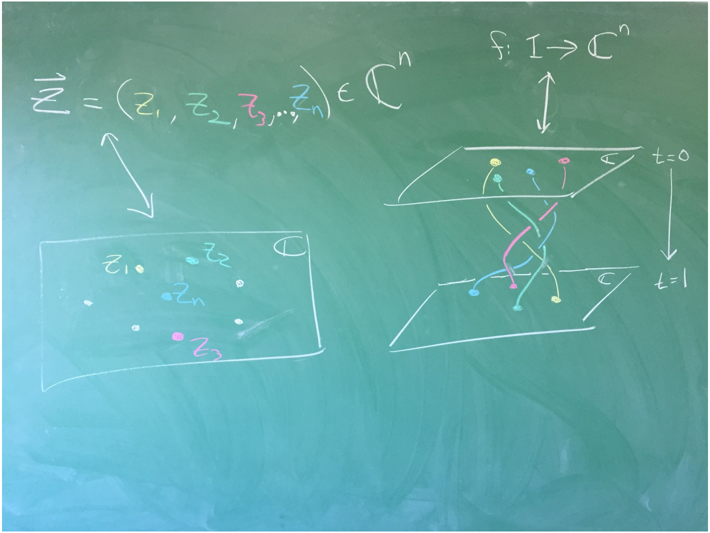
To find the braid group of a group $G$ acting on a space $X$ (1) remove the points with non-trivial stabilizers, (2) quotient by the resulting free $G$-action and (3) take the fundamental group of the quotient.

The name is derived from a classic example.

**Example (Artin's braid group)**

The braid group of $\textsc{Sym}_n$ acting on $\mathbb{C}^n$ by coordinate permutation is Artin's braid group.

# The Braid Arrangement



# Complexified Tits Cone

**Definition (Artin groups)**

The Artin group $Art(\Gamma)$ can be defined as the braid group of the action of $Cox(\Gamma)$ on the interior of its complexified Tits cone $\mathbb{C}Tits^{\circ}(\Gamma)$.

**Remark (Non-trivial Stablizers)**

The only points with non-trivial stabilizers are those in the hyperplanes so $Cox(\Gamma)$ acts freely on $\mathbb{C}Tits^{\circ}(\Gamma) \setminus \mathcal{H}$ where $\mathcal{H}$ is the union of the hyperplanes.

**Remark**

Rather than work with this space directly, we describe another space that is homotopy equivalent to it and much easier to understand and visualize.

# Adding Orientations

**Definition (Oriented $W$-Permutahedra)**

Let $P$ be a $W$-permutahedron. For every vertex $v$ in $P$ there is a unique opposite vertex. One can orient the 1-skeleton of $P$ using dot product with the vector from $v$ to its opposite as a height function. $P$ has as many orientations as it has vertices.

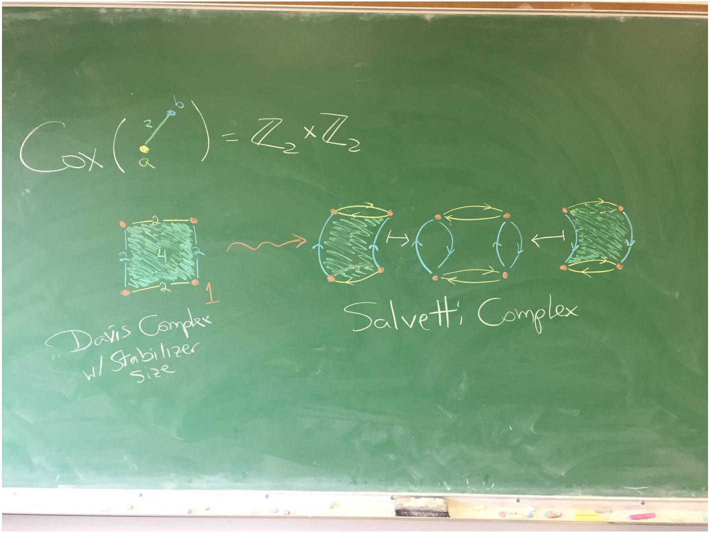
**Definition (Oriented Davis Complex)**

The oriented Davis complex replaces each permutahedron in the Davis complex with multiple copies, one for each possible orientation. It is also called the Salvetti complex $Salv(\Gamma)$.

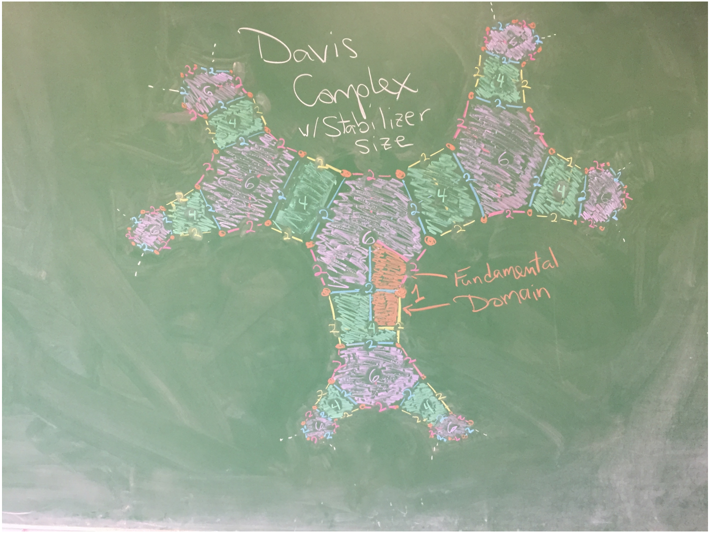
**Theorem**

*For each diagram $\Gamma$, $\mathbb{C}Tits^{\circ}(\Gamma) \setminus \mathcal{H}$ is homotopy equivalent to the Salvetti complex $Salv(\Gamma)$.*

# Salvetti Complex = Oriented Davis Complex



# A hyperbolic example: stabilizer size



# The Quotient Complex

**Remark (Quotient complex)**

The Coxeter group $W = Cox(\Gamma)$ acts freely on the oriented Davis complex and its quotient is a 1-vertex complex that has one oriented $W'$-permutahedron for each subset $S'$ of the standard generating set $S$ for which $W' = \langle S' \rangle$ is finite. Its fundamental group is the Artin group $Art(\Gamma)$.

**Remark (A classifying space?)**

The $K(\pi, 1)$ conjecture asserts that for every diagram $\Gamma$ this space is a classifying space for the group $Art(\Gamma)$.

# Summary

Here is a summary of the spaces and groups discussed today.

$$\begin{array}{ccccc}
\mathit{Cox}(\Gamma) & \curvearrowright & \mathit{Tits}^\circ(\Gamma) & \cong & \mathit{Davis}(\Gamma) \\
\mathit{Cox}(\Gamma) & \curvearrowright & \mathbb{C}\mathit{Tits}^\circ(\Gamma) \smallsetminus \mathcal{H} & \cong & \mathit{Salv}(\Gamma) \\
\mathit{Art}(\Gamma) & = & \pi_1(\mathit{Salv}(\Gamma)/\mathit{Cox}(\Gamma)) & &
\end{array}$$