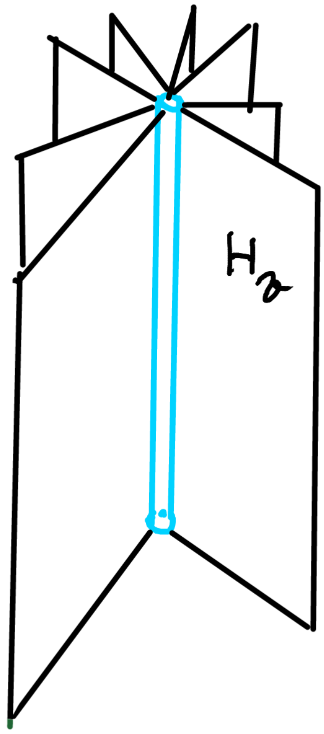
OPEN BOOK DECOMPOSITIONS

Remember:

$$\begin{array}{c} \mathbb{R}^3 - \{z\text{-axis}\} \\ \downarrow \vartheta \\ S^1 \end{array}$$



We will try to generalize:

Def: An open book decomposition of a closed oriented 3-mfd M is (L, π), where:

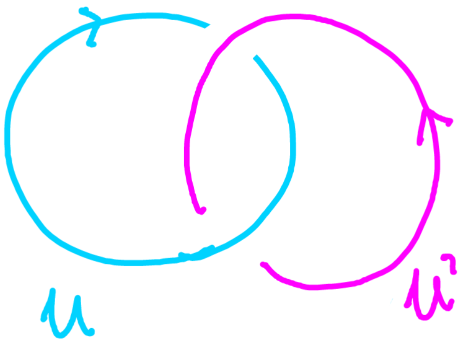
- $L \subset M$ link, called binding
- $\begin{array}{c} M - L \\ \downarrow \pi \\ S^1 \end{array}$ is a fibration so that $\Sigma_\vartheta = \overline{\pi^{-1}(\vartheta)}$ is a Seifert surface for L, called pages

e.g. $\mathcal{U} \subset S^3$ unknot.

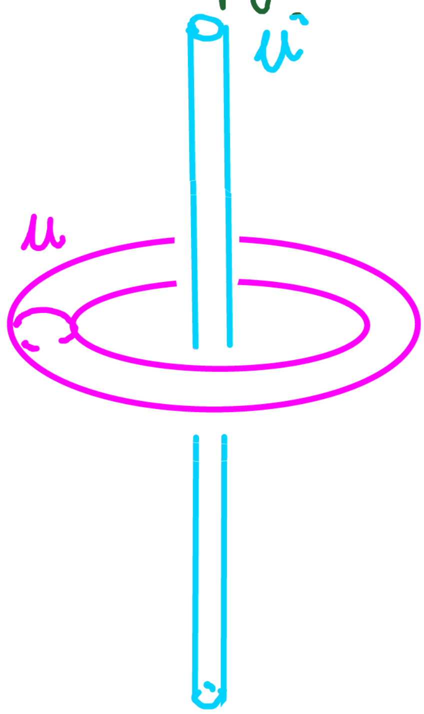
$S^3 - \mathcal{U} \cong \mathbb{R}^3 - \{z\text{-axis}\}$

binding, Σ_ϑ are discs

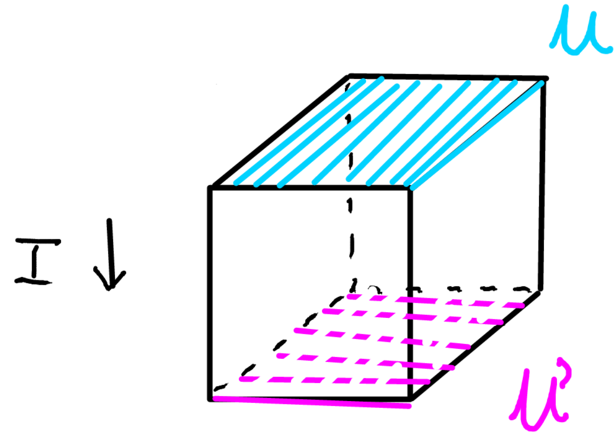
e.g. $\subseteq S^3$, H_+ Hopf link



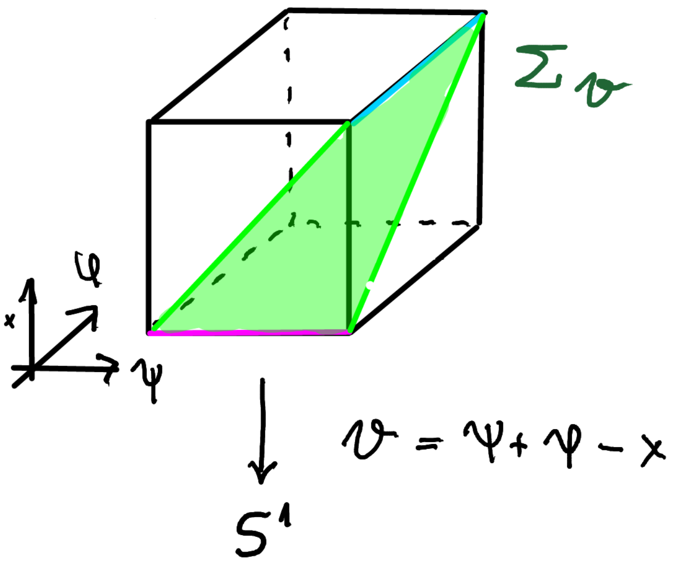
$\approx$

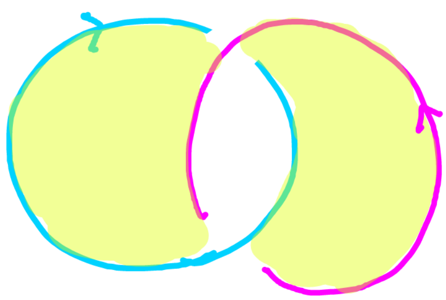


$\Rightarrow S^3 \setminus \mathcal{U} \cup \mathcal{U}' = T^2 \times I$



back & front and left & right } glued





or $S^3 = \{|z_1|^2 + |z_2|^2 = 1\} \subseteq \mathbb{C}^2$

$H_+ = \{ z_1 z_2 = 0\}$

$$\begin{array}{c} S^3 \setminus H_+ \\ \downarrow \frac{z_1 z_2}{|z_1 z_2|} \\ S^1 \end{array}$$

e.g. $p(z_1, z_2) : \mathbb{C}^2 \to \mathbb{C}$ polinomial

w/ $p(0,0) = 0$ & no other critical pt.

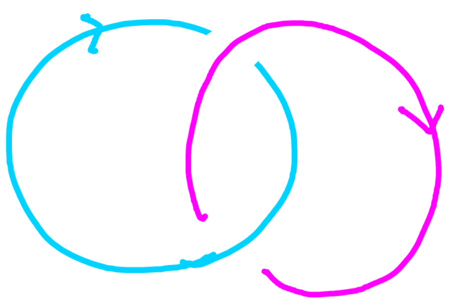
$$K_p := p^{-1}(0) \cap S^3$$

$$\begin{array}{c} S^3 \setminus K_p \\ \downarrow \frac{p(z_1,z_2)}{|p(z_1,z_2)|} \\ S^1 \end{array}$$ is an open book decomposition

HW Find a polynomial for

HW Find similar description for

H_- :



Thm (Alexander): Any M has an open book decomposition.

Proof (several proofs ...)

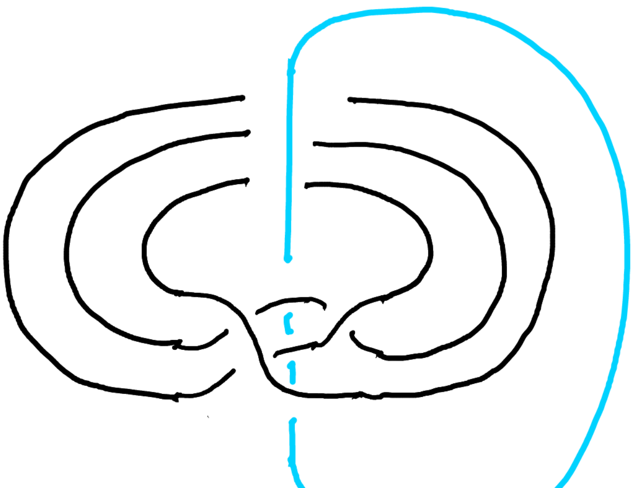
- M is 3-fold branched cover over S^3 branched over a link $K \subset S^3$:

$$\begin{array}{ccc} M & \supset & M \setminus g^{-1}(K) \\ \downarrow g & & \downarrow 3:1 \\ S^3 & \supset & S^3 \setminus K \end{array}$$

- put K in braid position $\pitchfork D^2_\theta$

$$\begin{array}{c} M \\ \downarrow g \end{array}$$

"pull back" (U, π)



$L := g^{-1}(U)$

$\pi' = \pi \circ g$

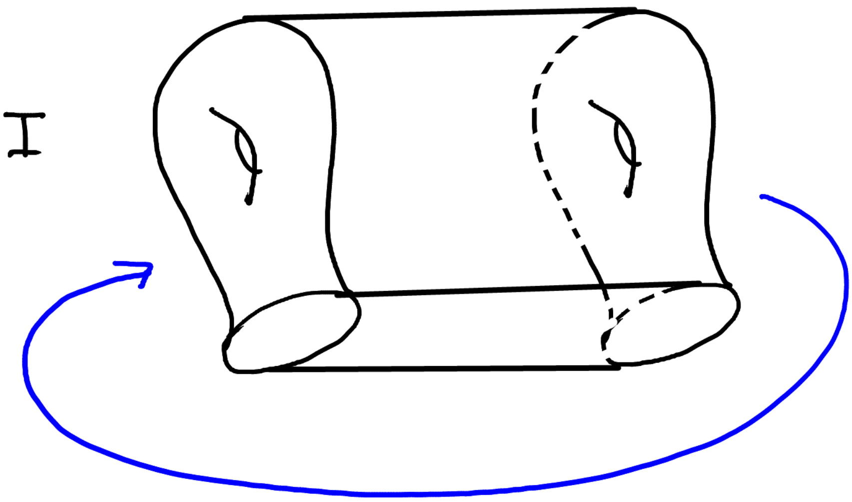
HW (L, π') gives an open book decomposition.

Abstract open book

(L,π) open book for M :
$$\begin{array}{c} M - L \\ \downarrow \pi \\ S^1 \end{array}$$

can be described by a monodromy :

$$\begin{array}{ccc} M \setminus S_0 & = & S \times I \\ \downarrow & & \\ S^1 \setminus \{0\} = I & & \end{array}$$



$h : S \to S$

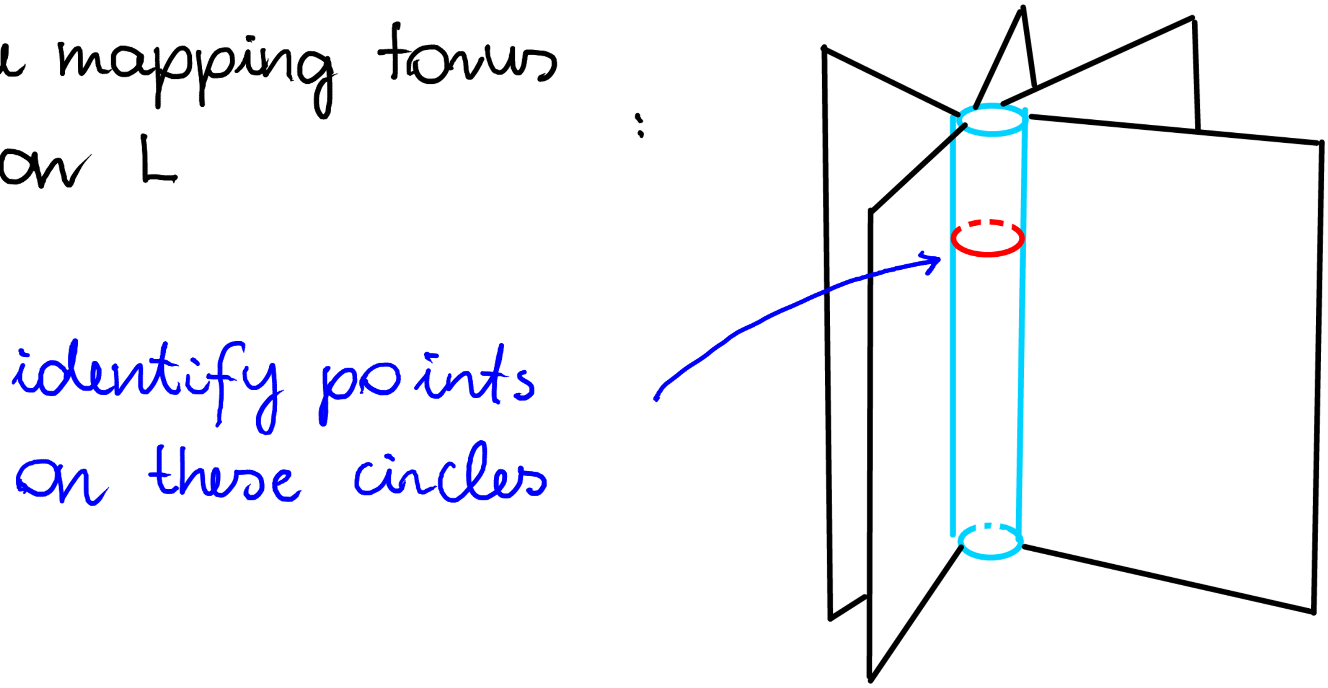
monodromy

$$h|_{N(\partial S)} = id$$

We can recover $M \setminus L$ as the mapping torus

$$M_h = {S \times I} \Big/ {(x,0) \sim (h(x),1)}$$

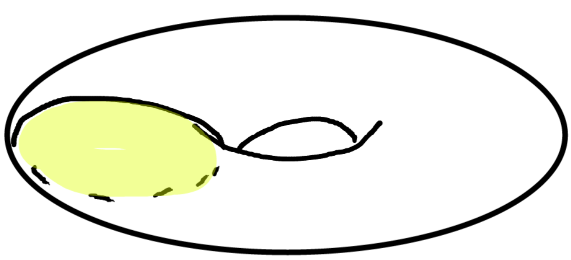
the mapping torus near L :



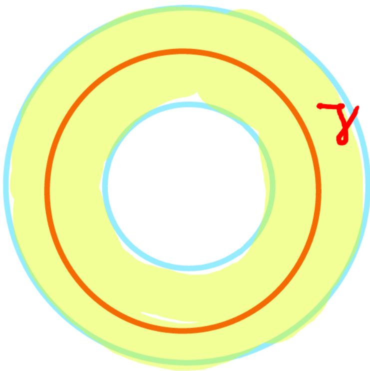
$$\boxed{M = {}^{S\times I}\Big/ {\scriptstyle \begin{array}{ll} (x,0)\sim(h(x),1) & x\in S \\ (x,t)\sim(x,t') & x\in\partial S,\ t,t'\in I \end{array}}}$$

$$(L,\pi)\Big/_{\text{isotopy}} \longleftrightarrow (S,h)\Big/_{\text{isotopy \& conjugation}}$$

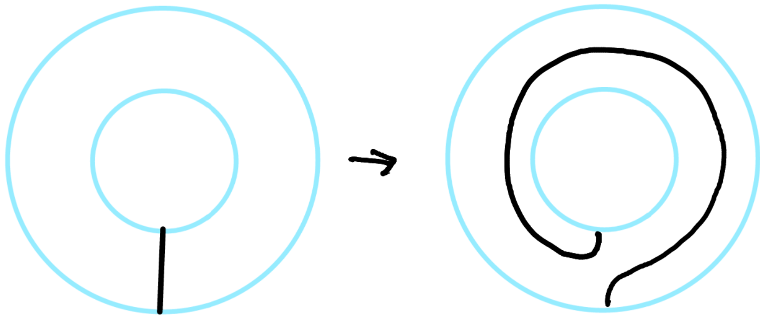
e.g.: $(D^2, id) \rightsquigarrow S^3$ w unknot



HW



(A, D_γ) right handed Dehn twist

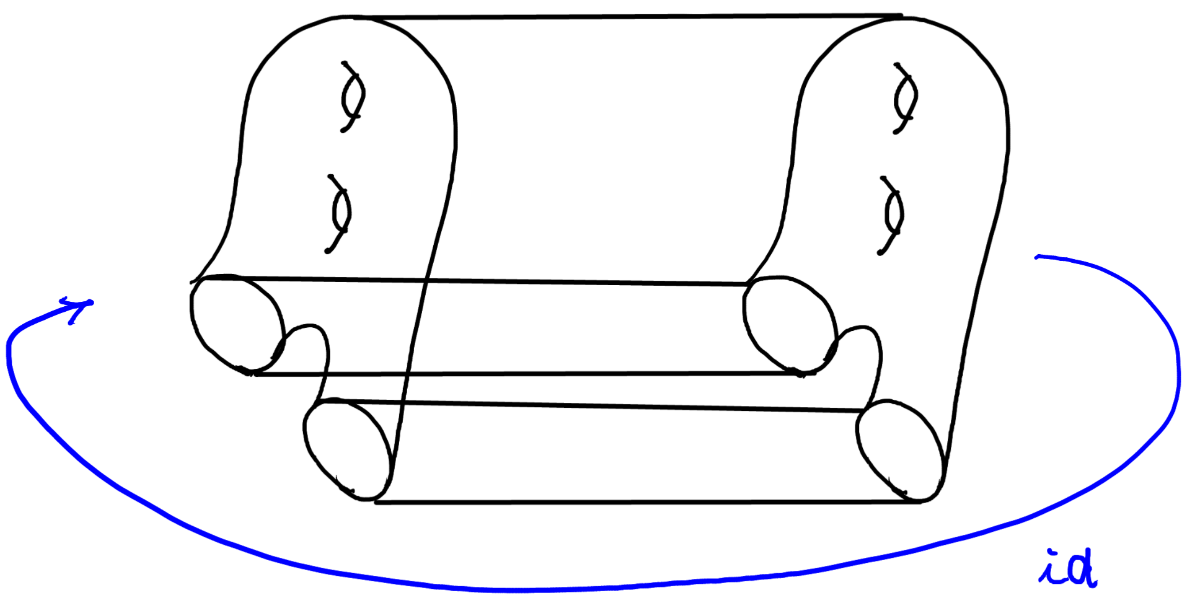


gives S^3 w/ H_+

HW (A, D_γ^{-1}) gives S^3 w/ H_-

HW What does (S, id) give?

genus g w/ n bdry components

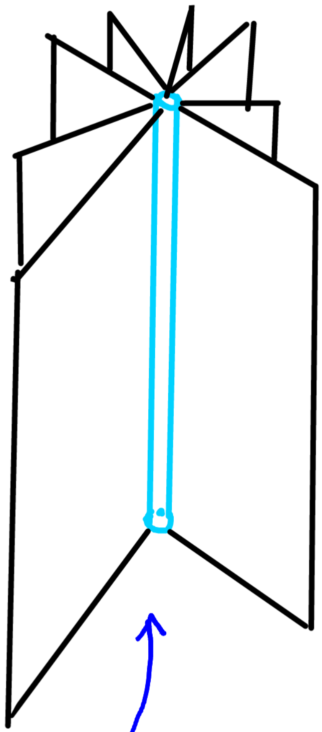


Connection to contact structures

Remember:

$$\begin{array}{c} \mathbb{R}^3 - \{z\text{-axis}\} \\ \big\downarrow \vartheta \\ S^1 \end{array}$$

ξ_{st} is "almost" $T\Sigma_\vartheta$



transverse knot

Def (Giroux 2000) The open book (L, π) is compatible w/ ξ if

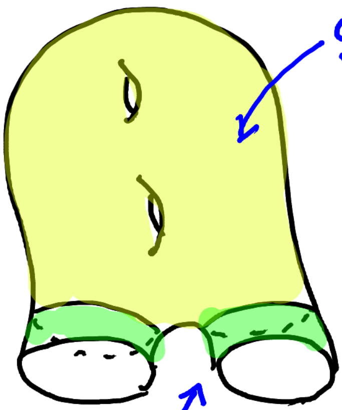
- L is transverse
- $\exists\ \alpha$ w/ $\xi = \ker \alpha$
  $d\alpha$ is an area form for Σ_ϑ $(\forall \vartheta)$

e.g. (U, π) supports ξ_{st}

HW (H_+, π) support ξ_{st}

Thm (Thurston – Winkelnkemper) Every open book admits a compatible contact structure.

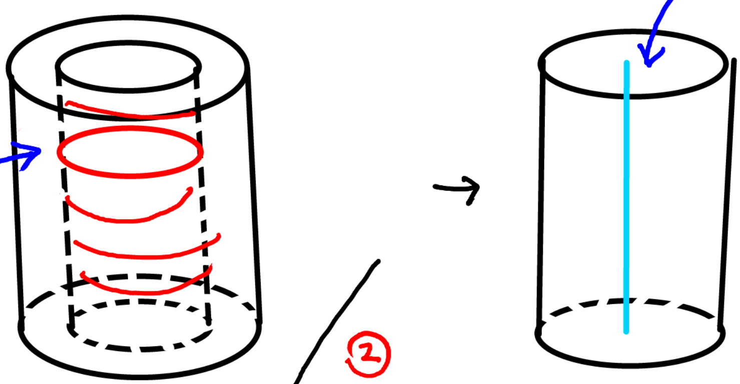
Proof:

$M =$ 

$\times I \Big/ \begin{array}{l} (x,0) \overset{①}{\sim} (h(x),1) \\ (x,t) \overset{②}{\sim} (x,t') \quad x \in \partial\Sigma \end{array}$

assume $h|_{N(\partial S)} = id$

$N(\partial S)/\overset{①}{\sim}$:



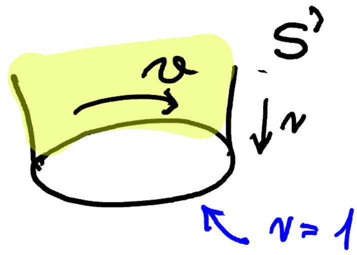
$\Rightarrow M = \underbrace{S' \times I/\overset{①}{\sim}}_{M_{h|S'}} \cup$ solid tori

We will construct contact structure on them separately

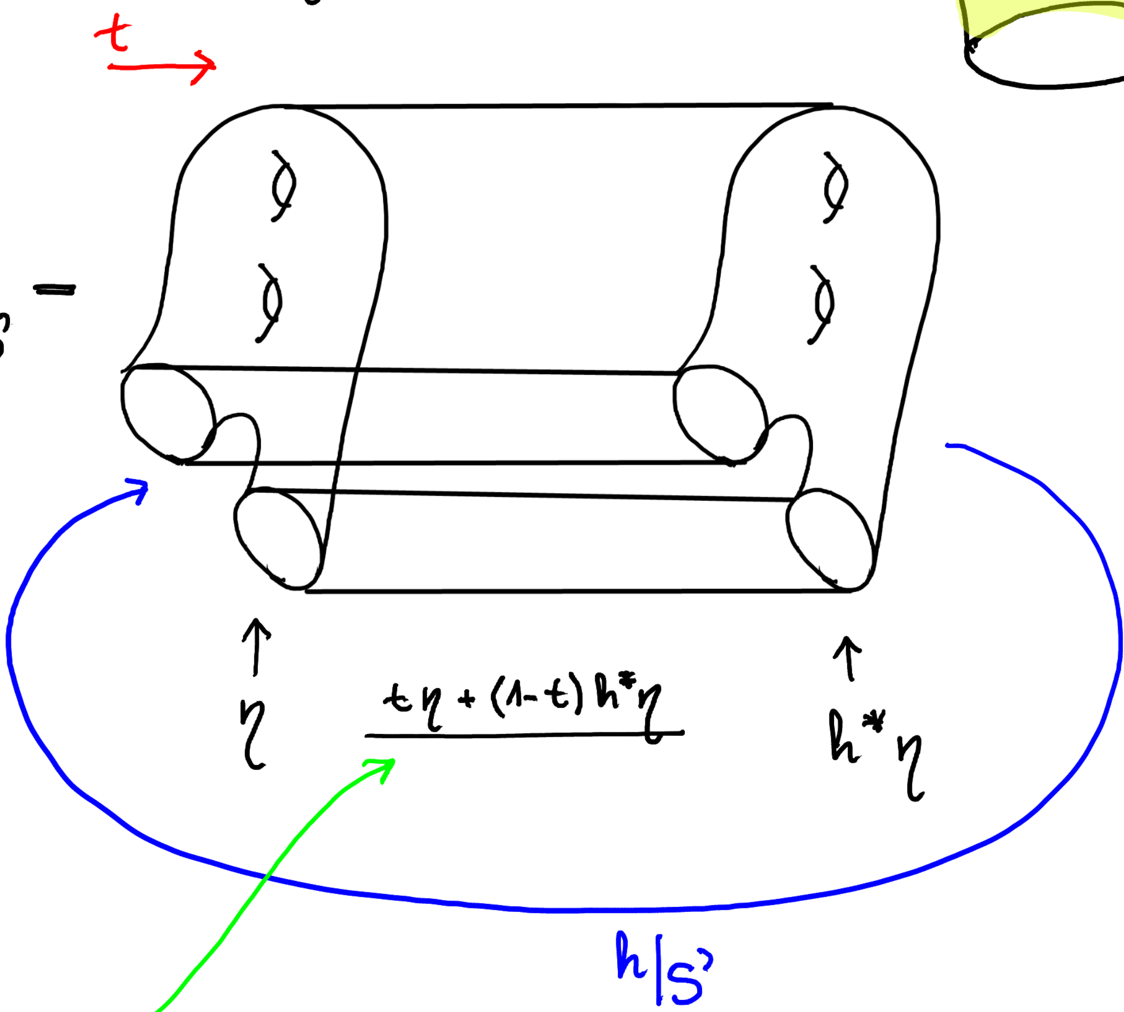
• $M_{h|S'}$: HW There is a 1-form η on S' s.t.

(1) $d\eta$ is a volume form on S'

(2) $\eta = r\,d\vartheta$ near $\partial S'$:



$M_{h|S'} =$



HW $d(\quad)$ are all volume forms

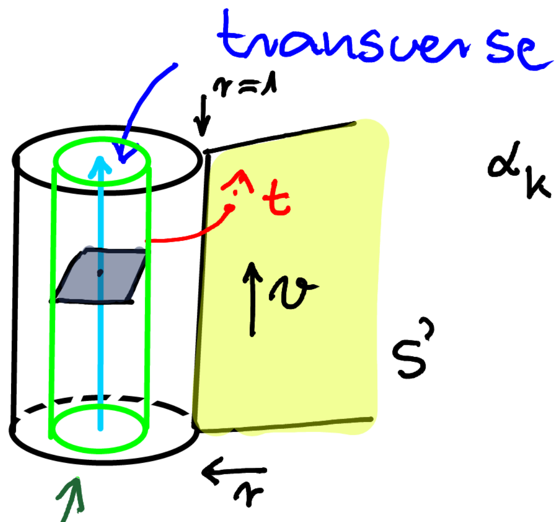
$\rightsquigarrow$ 1-form α' on $M_{h|S'}$ w/

$\alpha' = r\,d\vartheta$ near $\partial S' \times \{t\}$ $(\forall\, t)$

HW For $K \gg 0$ $\alpha_K = \alpha' + K\,dt$ is a contact form on $M_{h|S'}$

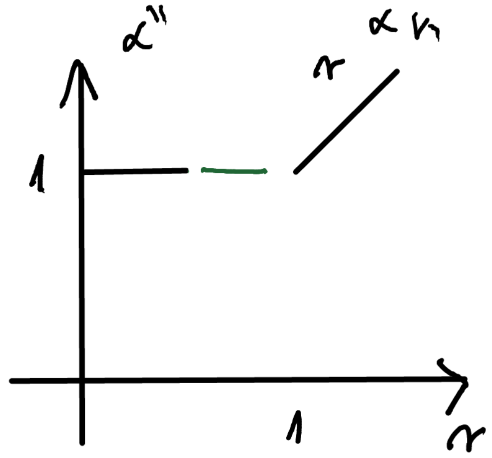
• solid tori :

$$\alpha_K = r\,d\vartheta + K\,dt$$

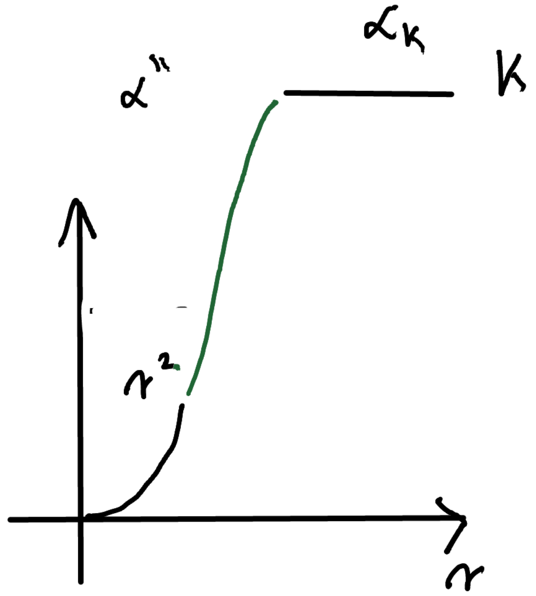


$$\alpha'' = d\vartheta + r^2\,dt$$

$d\vartheta$:



dt :



HW Can choose smooth functions to connect α'' & $\alpha_K \big|_{r \geq 1+\varepsilon}$. ∎

HW (Uniqueness): If ξ & ξ' are both compatible w/ the same open book $\Rightarrow$ ξ is isotopic to ξ'.

$$\left\{\begin{matrix}\text{open books}\\ \text{of } M\end{matrix}\right\}\Big/\text{isotopy} \xrightarrow{X} \left\{\begin{matrix}\text{contact}\\ \text{structures on } M\end{matrix}\right\}\Big/\text{isotopy}$$

Thm. (Giroux '00):

- X is surjective
- Any two open books compatible w/ the same contact structure are related by stabilisations

define later

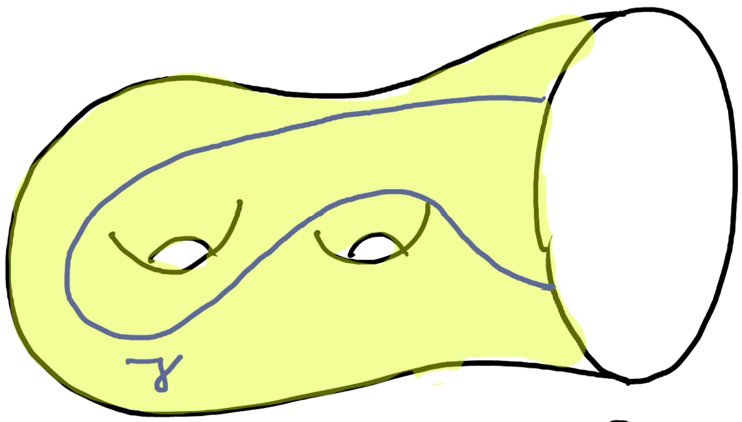
Giroux correspondence

$$\left\{\begin{matrix}\text{open books}\\ \text{of } M\end{matrix}\right\}\Big/\begin{matrix}\text{isotopy}\\ +\\ \text{stabilisation}\end{matrix} \overset{\tilde{X}}{\longleftrightarrow} \left\{\begin{matrix}\text{contact str.}\\ \text{on } M\end{matrix}\right\}\Big/\text{isotopy}$$

! Giroux correspondence gives a completely geometric/combinatorial description of contact structures!

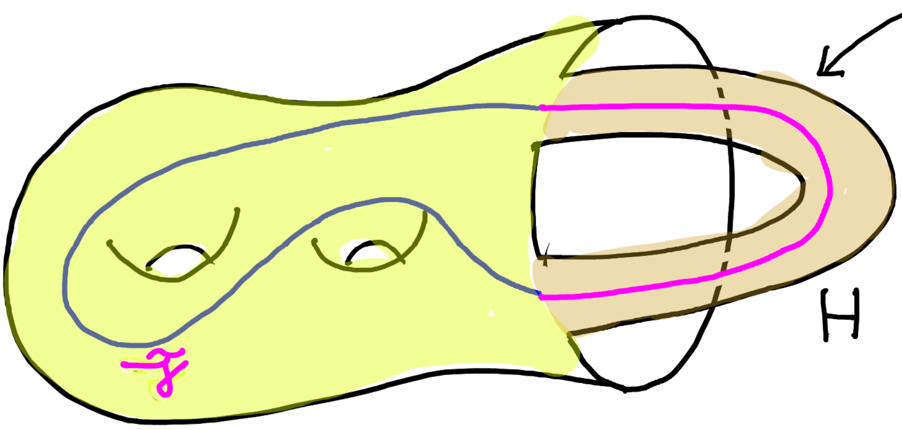
Stabilisation

(S, h) abstract open book for (M, ξ)



S

$\gamma \hookrightarrow S$
properly
embedded arc



1-handle
attached
along
$\partial\gamma$

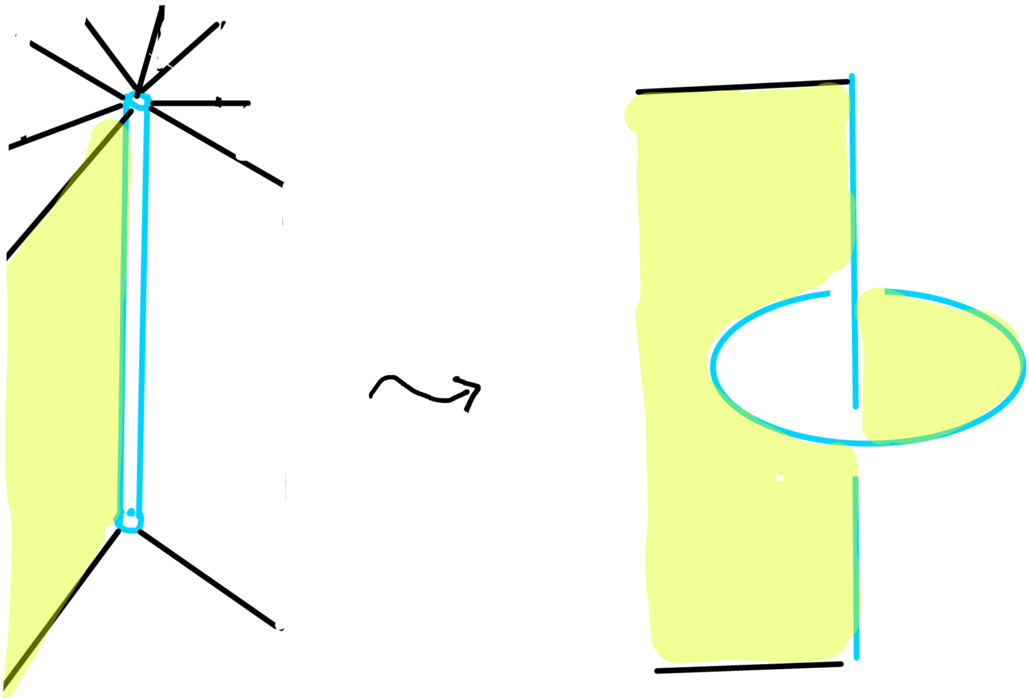
$S' = S \cup H$

$\tilde{\gamma} = \gamma \cup (\text{core of } H)$

$h' := h \circ R_{\tilde{\gamma}}$

Right-handed Dehn twist about γ $\rightsquigarrow (S', h')$

in M:



(HW) Verify the above picture.

(HW) Try to draw the picture in general.

(HW) Show that $(\hat{S}, \hat{h})$ gives back M.

(HW) Show that (S', h') is compatible w/ (M, ξ).

(HW) Construct open books w/ torus-knot binding.

Generalised Braids

def A knot K is braided w.r.t. an open book if $K \pitchfork S_\vartheta \;\; (\forall \vartheta)$

$$S_\vartheta \text{ ``}\pi^{-1}(\vartheta)\text{''}$$

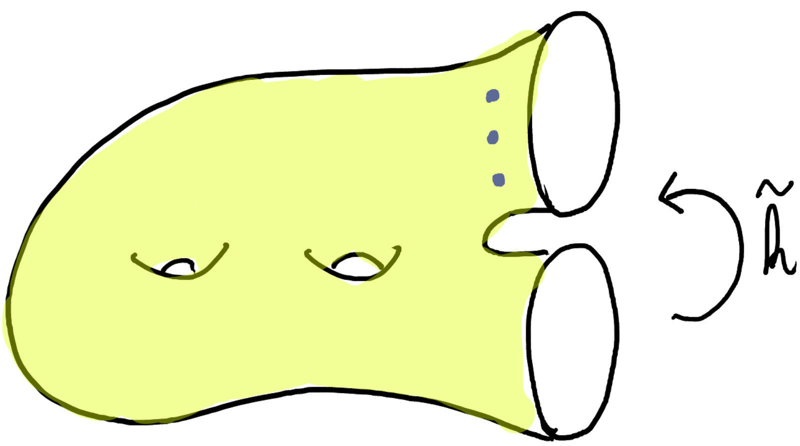
$(\Leftrightarrow \pi|_K \to S^1$ is a covering map$)$

Generalisation of Markov Thm.

Thm (Skora '92)

Any knot can be put in braid position w.r.t. any open book

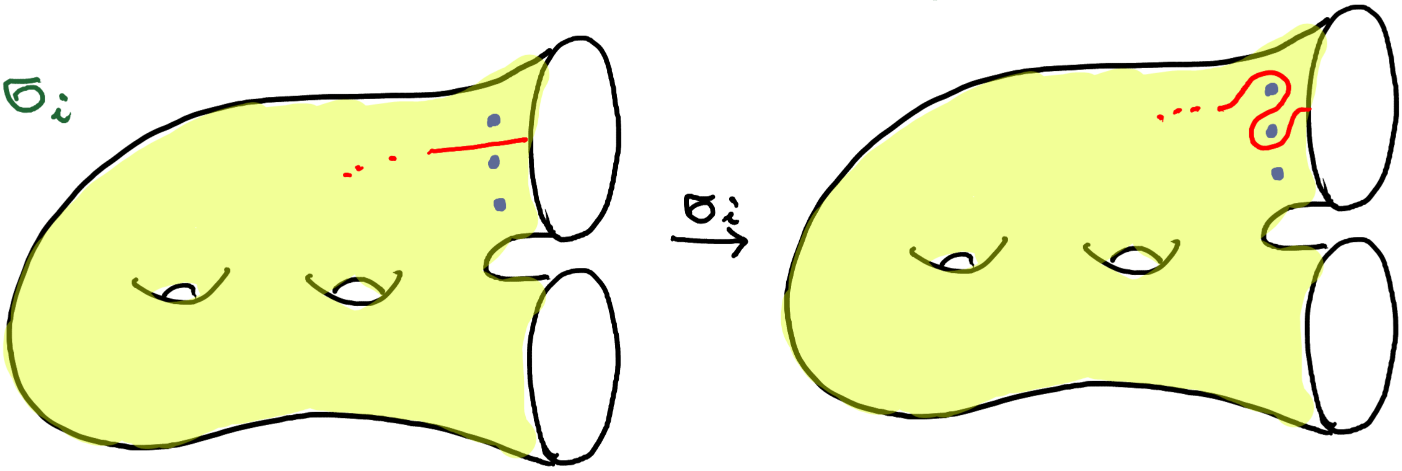
$\Rightarrow$ they can be described w/ mapping class groups:



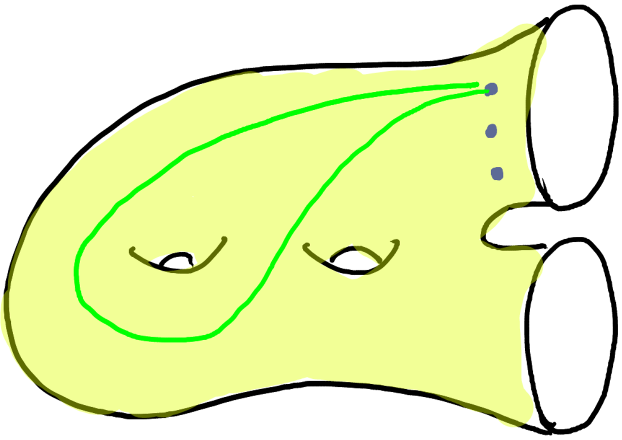
braids w.r.t (S, h) / braid isotopy $\longleftrightarrow (S, p) \circlearrowleft \tilde{h} \xrightarrow[\text{functor}]{\text{forgetfull}} h$ / isotopy rel. ∂S & p

generators of the mapping class group

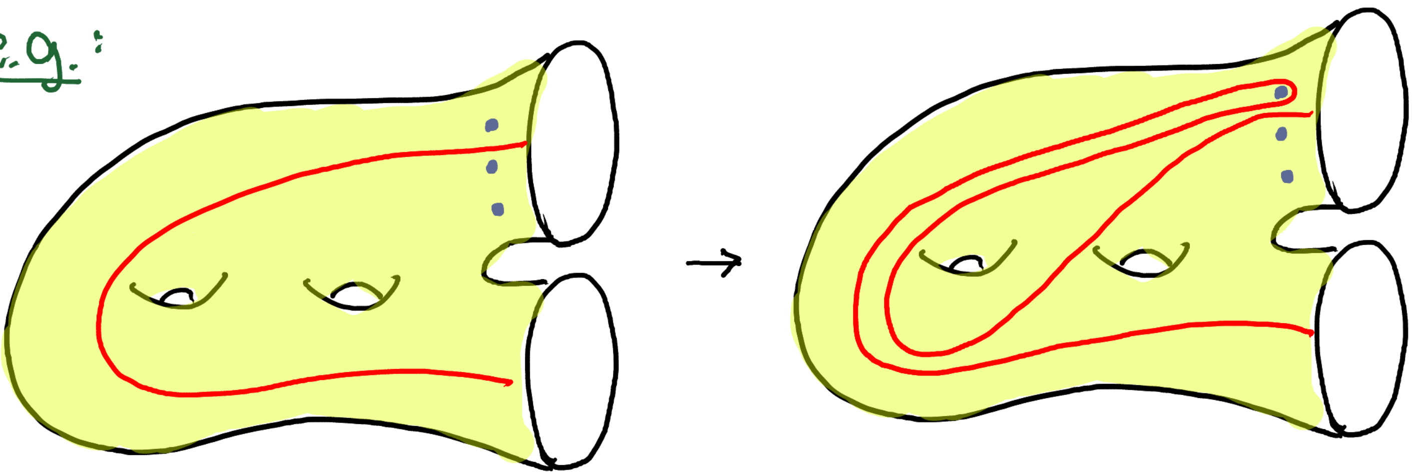
σ_i



fingermoves



e.g.:



generalised Alexander Thm:

Thm (Sundheim '93): Two knots given as braids w.r.t. an open book are isotopic
$\Leftrightarrow$ they are related by braid isotopies & Markov moves

For transverse knots:

Thm (Pavelsku '08) ξ supported by (L, π)

- Any transverse knot can be put in braid position

- Two transverse knots given as braids are transverse isotopic
$\Leftrightarrow$ they are related by braid isotopies & positive Markov moves

$\rightsquigarrow$ can define minimal braid index w.r.t. any open book.

$$b_{(L,\pi)}$$

HW Prove that for any knot K there is an open book s.t.

$$b_{(L,\pi)}(K) = 1$$

$\rightsquigarrow$ the minimal genus of such an open book gives another knot invariant

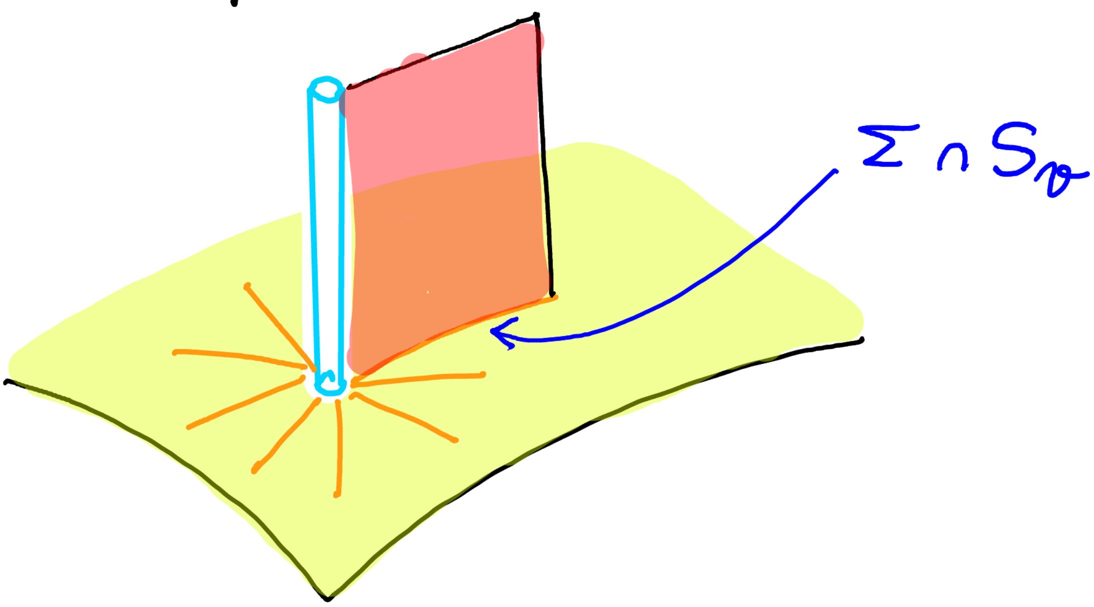
$$g(K) = \min \left\{ g(S) : \begin{array}{l} K \text{ is a 1-braid} \\ \text{w.r.t. to the open book} \\ (S, h) \end{array} \right\}$$

Little is known about these invariants...

Open book foliations

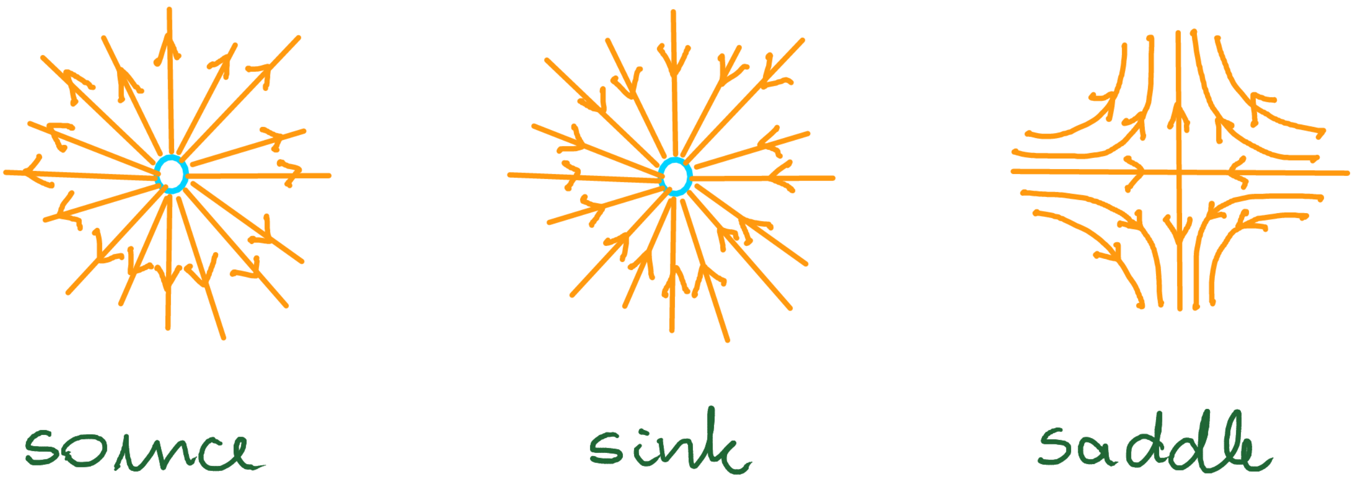
Ito - Kawamuro generalised braid foliations for open books:

$\Sigma \hookrightarrow M$ w/ open book (L, π), $S_\vartheta = \pi^{-1}(\vartheta)$



$\rightsquigarrow$ open book foliation on Σ

generically only w/ finitely many singularities of the types



Ito-Kawamuro understood

- "moves" of OB foliation induced by isotopies of Σ

- the effect of stabilisation of an ob on the ob foliation

Remember: in (D^2, id) K is given by $(D^2, P) \circlearrowleft \tilde{h} \longleftrightarrow w \in B_n$ braid word

$|P|$ algebraic length

$$sl(K) = n + a(w)$$

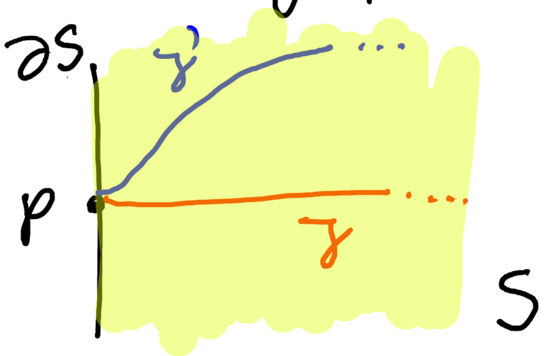
These are all invariants of $\tilde{h}$.

Thm (Ito-Kawamuro): K is given by $(S, P) \circlearrowleft \tilde{h}$ then $sl(K)$ can be computed in terms of $\tilde{h}$.

$\rightsquigarrow$ when $sl(K) = -\chi(\Sigma)$? in simple cases.

Recognising overtwisted discs

A properly embedded arc γ is **to the left** of another arc γ' at their common starting pt. p if after putting them in minimally intersecting position we have



$\gamma \prec \gamma'$

Thm (Honda-Kazez-Matic)

If $\exists\ \gamma$ not fixed by h s.t. $h(\gamma)$ is to the left from $\gamma \Rightarrow (S, h)$ is overtwisted.

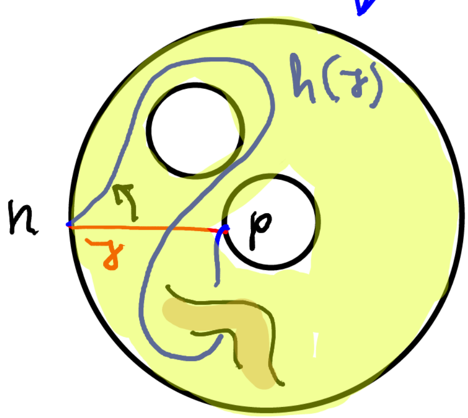
Thm (Honda-Kazez-Matic)

ξ is overtwisted $\Leftrightarrow \exists$ such γ for some open book compatible w/ ξ

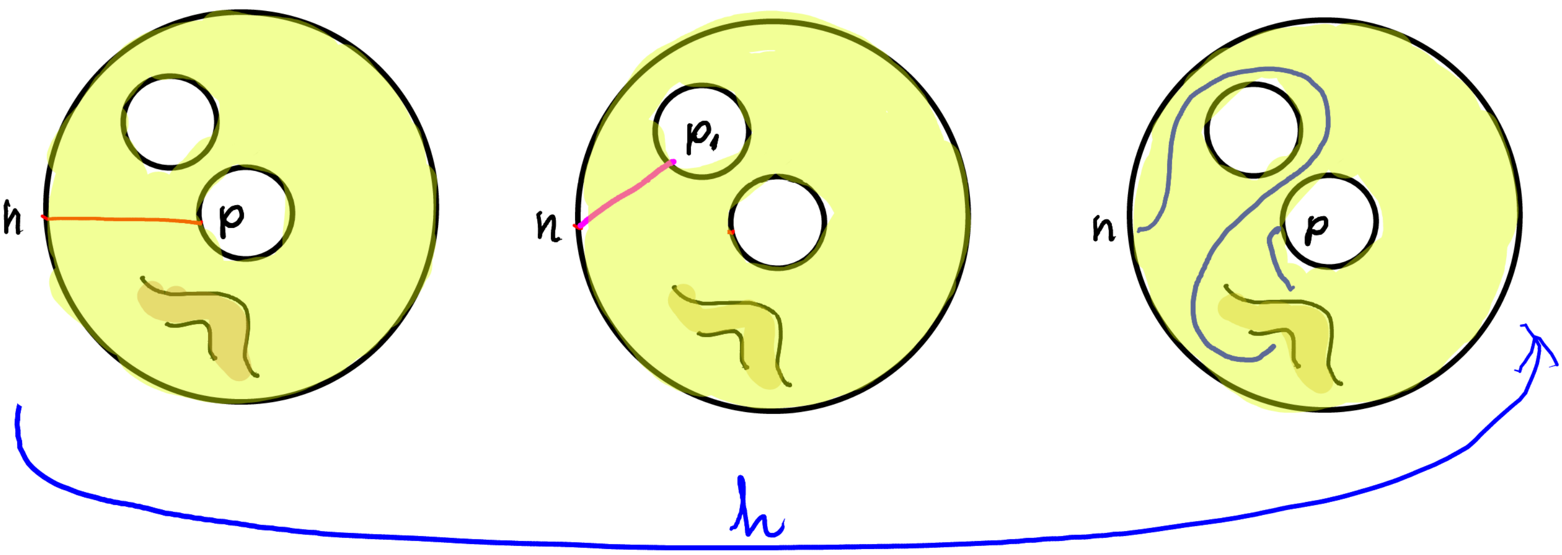
Proof (only the first) Using open book foliations

twice punctured torus

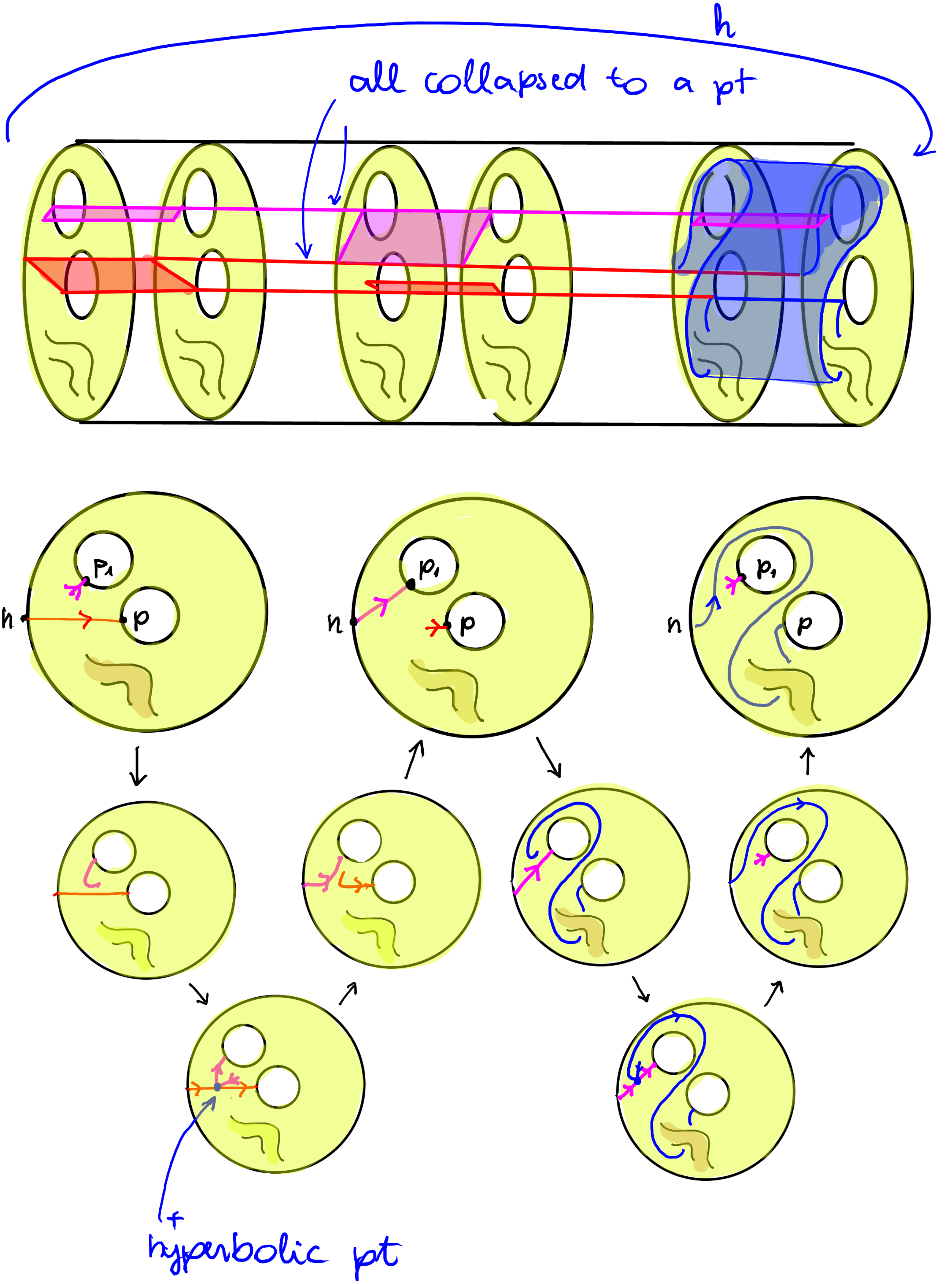
- suppose $h(\gamma)$ is to the left of γ at n

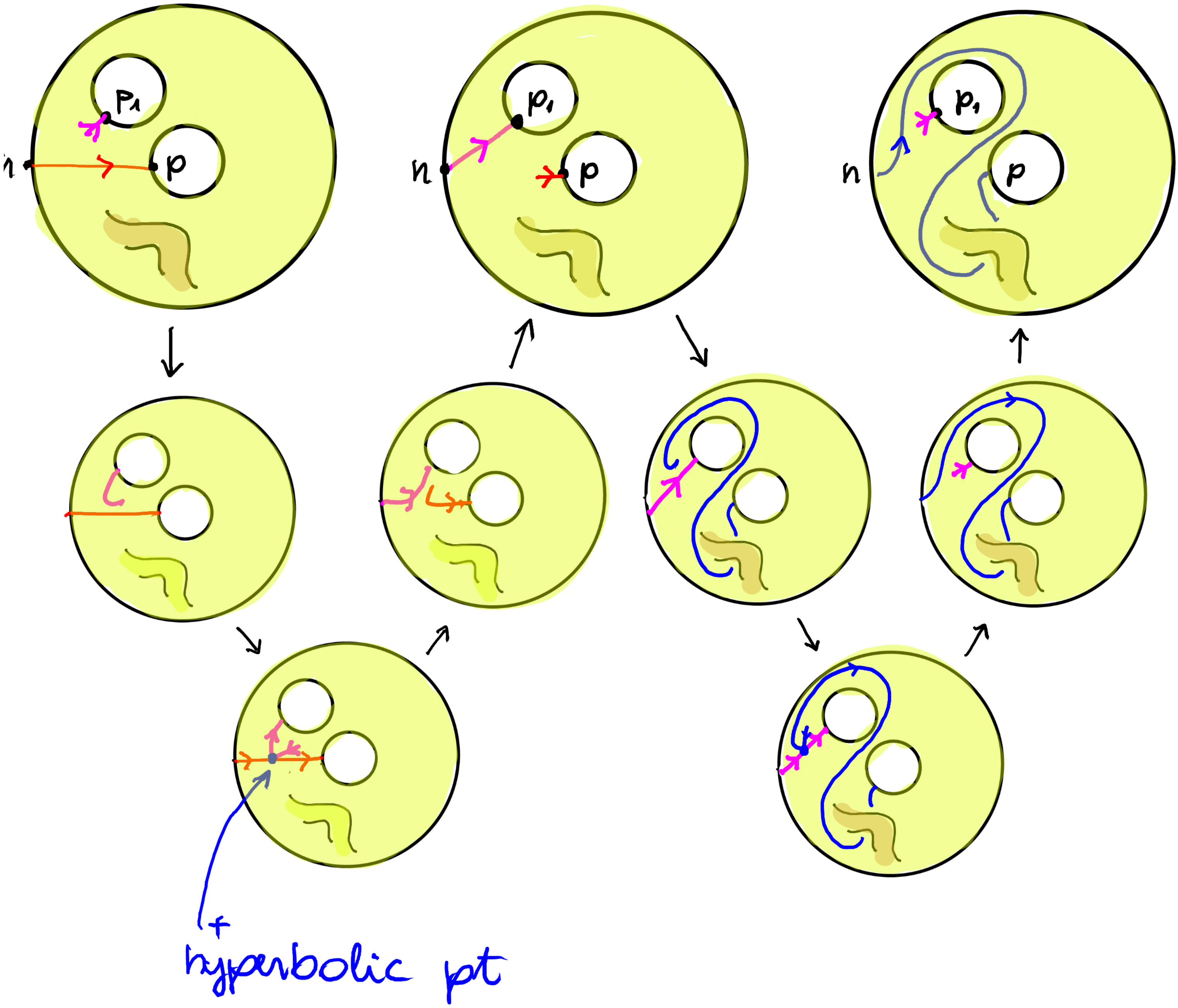


- choose arcs γ_i s.t.
  - → $\gamma_0 = \gamma$, $\gamma_n = h(\gamma)$
  - → γ_i is an arc $n \to p_i$ ($p_0 = p_n$ & all other p_i's are different)
  - → γ_{i+1} is to the left from γ_i at n

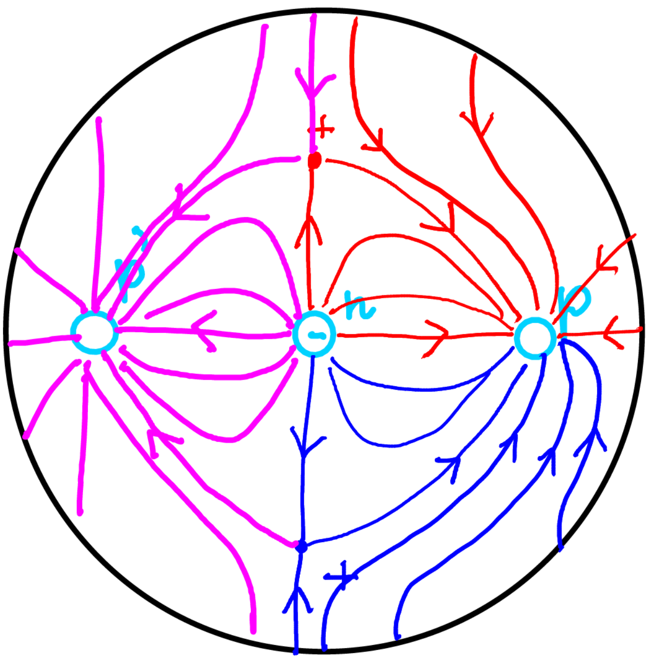


want to define an embedded disc D in (M, ξ) w/ $sl(\partial D) > -1$


h
all collapsed to a pt
p1
p
n
hyperbolic pt



the foliation on D:



$e_- = 1$

$e_+ = 2$

$h_+ = 2$

$h_- = 0$

$\Downarrow$

$sl(\mathcal{U}) = +1$

:)